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Abstract: In this work we present an extension of the continuous interior penalty method of Douglas and Dupont [Lecture Notes in Phys., Vol. 58 (1976) 207] to the incompressible Navier-Stokes equations. The method consists of a stabilized Galerkin formulation using equal order interpolation for pressure and velocity. To counter instabilities due to the pressure/velocity coupling, or due to high local Reynolds number, we add a stabilization term giving L^2-control of the jump of the gradient over element edges (faces in 3D) to the standard Galerkin formulation. Boundary conditions are imposed in a weak sense using a consistent penalty formulation due to Nitsche. We prove energy type a priori error estimates independent of the local Reynolds number and give some numerical examples recovering the theoretical results.

Key-words: Oseen equation, interior penalty, gradient jump, minimal stabilized method

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Stabilisation par pénalisation intérieur d'une méthode d'éléments finis pour les équations de Navier-Stokes incompressible

Résumé : Dans ce travail on introduit une extension de la méthode de pénalisation intérieure due à Douglas et Dupont [Lecture Notes in Phys., Vol. 58 (1976) 207] au cas des équations de Navier-Stokes incompressible. La méthode est obtenue après stabilisation de la méthode de Galerkin avec le même ordre d'interpolation pour la vitesse et la pression. Les instabilités issues du couplage vitesse/pression, ou liées à des Reynolds locaux élevés, sont traitées en rajoutant à la formulation de Galerkin un terme stabilisant qui donne contrôle L^2 du saut du gradient sur les arêtes (faces en 3D). Les conditions aux limites sont imposées faiblement en utilisant la formulation pénalisée de Nitsche. On montre des estimations d'erreur dans la norme de l'énergie indépendantes du nombre de Reynolds local. Les exemples numériques reproduisent les résultats théoriques.

Mots-clés : Équation d'Oseen, pénalisation intérieure, saut de gradient, méthode stabilisée minimale

1 Introduction

The construction of finite element methods for the incompressible Navier-Stokes equations that are robust and accurate for a wide range of Reynolds numbers remains a challenging problem. The standard Galerkin method requires the fulfillment of the *inf-sup* or Babuska-Brezzi condition, which leads to the need of using mixed interpolations (see [5, 21]). Obviously, from the computational point of view, it is more practical to use equal order interpolation for the velocity and pressure spaces. This requires that stability is imposed in some other fashion. Usually by resorting to *stabilized* finite element properties where some terms are added to enhance the stability properties of the method. In addition to the satisfaction of the *inf-sup* condition the method must be stable with respect to the convective terms, and also give sufficient control of the incompressibility condition.

A popular approach has been to stabilize both the velocities and the pressure using the streamline upwind Petrow-Galerkin method (SUPG). This method was first analyzed for the Navier–Stokes equations in a velocity vorticity formulation in [26], and then in pressure velocity formulation in [23, 20, 29]. The SUPG method owes it success to the unified treatment of velocities and pressures. It allows for a priori error estimates that are independent of the Reynolds number and has been used extensively in practice with good results. However, the SUPG method has some undesirable features:

- artificial boundary conditions on velocities and pressure are introduced;
- artificial non-symmetric terms are introduced;
- the least squares term introduces artificial pressure-velocity couplings in the matrix;
- the least squares term makes mass lumping impossible and time stepping awkward; to stay consistent one has to ressort to a space-time finite element approach using discontinuous approximation in time;
- how to use mixed finite elements in combination with SUPG is still unclear.

To overcome these disadvantages some new stabilization techniques have been developed such as the projection method proposed by Codina [15, 14], the subgrid viscosity method or local projection method proposed by Guermond [22] and Becker and Braack [1] and the polynomial pressure projection method by Bochev and Dohrmann reported in [2]. More recently the edge stabilization method of Douglas and Dupont [16] was revived as an alternative. It was shown in [11, 10] that the method stabilizes both instabilities due to dominating convection and instabilities due to the velocity-pressure coupling. Moreover it was shown in [7] how this method provides a natural link between conforming and non-conforming finite element methods. It was used in [12] to provide a Reynolds number independent stabilized formulation for the classical non-conforming P_1 Crouzeix-Raviart approximation for the velocities combined with elementwise constant pressures.

In this paper we give the first extension of the method to the linearized Navier-Stokes equations. We point out that the proposed method circumvents all the above mentioned inconveniences of the SUPG method at the price of some added couplings in the Jacobian matrix; the bandwidth of the system matrix doubles in two space dimensions and triples in three space dimensions. The formulation allows for general unstructured meshes. In this paper we only treat the case of equal order interpolation for pressures and velocities. For the case of similar stabilization strategies for element pairs satisfying the LBB-condition we refer to [8].

For the linearized Navier–Stokes equations (a.k.a. Oseen's equations) we prove a (quasi) optimal a priori error estimate in the energy norm independent of the Reynolds number if the solution is sufficiently regular. We then consider physically realistic global regularities for the different Reynolds number regimes. An error estimate for the L^2-norm of the velocity is then proved. Finally we study the performance of the numerical scheme on some linear model cases in two and three space dimensions. Somme of these results have been already announced as a brief note in [9].

2 A finite element method for the Oseen's equation

We consider, in $\Omega \subset \mathbb{R}^d$ with boundary $\partial\Omega$, the problem of solving

$$\left\{ \begin{aligned} \sigma \mathbf{u} + \boldsymbol{\beta} \cdot \nabla \mathbf{u} + \nabla p - 2\nu \nabla \cdot \boldsymbol{\varepsilon}(\mathbf{u}) &= \mathbf{f} \quad \text{in} \quad \Omega, \\ \nabla \cdot \mathbf{u} &= 0 \quad \text{in} \quad \Omega, \\ \mathbf{u} &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \right. \tag{1}$$

where $\mathbf{u} \in [H_0^1(\Omega)]^d \cap H_0(\mathrm{div};\Omega)$, $\boldsymbol{\beta} \in W^{1,\infty}(\Omega) \cap H_0(\mathrm{div};\Omega)$, $p \in L_0^2(\Omega)$, $\mathbf{f} \in [L^2(\Omega)]^d$ is a given source term, σ and ν are bounded positive functions. By $H_0(\mathrm{div};\Omega)$ we denote the functions in $[L^2(\Omega)]^d$ such that $\nabla \cdot \mathbf{u} = 0$. We denote the L^2-scalar product over Ω by $(\cdot,\cdot)$ and the corresponding norm by $\|\cdot\|_{0,\Omega}$. The $H^s(\Omega)$ norm will be denoted by $\|\cdot\|_{s,\Omega}$.

The weak formulation of problem (1) reads: find $(\mathbf{u}, p) \in [H_0^1(\Omega)]^d \times L_0^2(\Omega)$ such that

$$\left\{ \begin{aligned} a(\mathbf{u},\mathbf{v}) + b(p,\mathbf{v}) &= (\mathbf{f},\mathbf{v}) \\ b(q,\mathbf{u}) &= 0 \end{aligned} \right. \quad \forall (\mathbf{v}, q) \in [H_0^1(\Omega)]^d \times L_0^2(\Omega), \tag{2}$$

where

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) &\stackrel{\text{def}}{=} (\sigma\, \mathbf{u},\, \mathbf{v}) + (\boldsymbol{\beta} \cdot \nabla \mathbf{u},\, \mathbf{v}) + 2(\nu \boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v})), \\ b(p,\mathbf{v}) &\stackrel{\text{def}}{=} -(p, \nabla \cdot \mathbf{v}). \end{aligned} \tag{3}$$

The well-posedness of this problem follows by the Lax-Milgram lemma applied in the space $[H_0^1(\Omega)]^d \cap H_0(\mathrm{div};\Omega)$ (see, for instance, [21]).

Let $\mathcal{T}_h$ denote a triangulation of the domain Ω without hanging nodes. For each $K \in \mathcal{T}_h$, we let

$$h_K \stackrel{\text{def}}{=} \max_{e \subset \partial K} h_e,$$

with h_e the diameter of the edge e. The interior of a triangle K will be denoted by $\overset{\circ}{K}$. Moreover we will assume that the mesh is regular the sense that

- (local shape regularity)
$$\frac{h_K}{\mathrm{diam}(K)} < C, \quad \forall K \in \mathcal{T}_h,$$
where $\mathrm{diam}(K)$ stands for the diameter of the largest inscribed ball in K and C is a fixed constant;

- (local quasi uniformity) for any two elements $K, K' \in \mathcal{T}_h$ having at least one common node there holds $h_K \leq \rho h_{K'}$ where $\rho > 0$ is a parameter depending on the mesh regularity.

For the error analysis, we shall use the following trace inequality

$$\|v\|^2_{0,\partial K} \leq C\Big(h_K^{-1}\|v\|^2_{0,K} + h_K\|v\|^2_{1,K}\Big) \quad \forall v \in H^1(K), \tag{4}$$

where C is a generic constant independent of h_K (for a proof see [28, p. 26]).

For a given piecewise continuous function φ, the jump $[\![\varphi]\!]_e$ over a edge e is defined by

$$[\![\varphi]\!]_e(\mathbf{x}) \stackrel{\mathrm{def}}{=} \begin{cases} \lim\limits_{t\to 0^+} (\varphi(\mathbf{x}+t\mathbf{n}_e) - \varphi(\mathbf{x}-t\mathbf{n}_e)), & \text{if} \quad e \not\subset \partial\Omega, \\ 0, & \text{if} \quad e \subset \partial\Omega, \end{cases}$$

where $\mathbf{n}_e$ is a normal unit vector on e and $\mathbf{x} \in e$.

In this paper we let V_h^k denote the standard space of continuous functions of piecewise poynomial order $k \geq 1$,

$$V_h^k \stackrel{\mathrm{def}}{=} \left\{ v \in H^1(\Omega) \,:\, v_{|K} \in P_k(K), \quad \forall K \in \mathcal{T}_h \right\}.$$

and $H^2(\mathcal{T}_h)$ the space of piecewise H^2 functions

$$H^2(\mathcal{T}_h) \stackrel{\mathrm{def}}{=} \left\{ v : \Omega \longrightarrow \mathbb{R} \,:\, v_{|K} \in H^2(K), \quad \forall K \in \mathcal{T}_h \right\}.$$

For the velocities we will use the space $[V_h^k]^d$ and for the pressure we will use $Q_h^k = V_h^k \cap L_0^2(\Omega)$. In the sequel, we let π_h denote the standard L^2-projection operator onto the finite element spaces, and we make no notational difference between the projection onto the velocity and pressure spaces. We also introduce a piecewise linear approximated velocity $\boldsymbol{\beta}_h \in [V_h^1]^d$ such that

$$\|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_{0,\infty,K} \leq C h_K \|\boldsymbol{\beta}\|_{1,\infty,K}, \quad \forall K \in \mathcal{T}_h. \tag{5}$$

Denoting the product space $W_h^k \stackrel{\mathrm{def}}{=} [V_h^k]^d \times Q_h^k$ our finite element method reads: find $(\mathbf{u}_h, p_h) \in W_h^k$ such that

$$\mathbf{A}\big((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)\big) = \mathbf{F}\big((\mathbf{v}_h, q_h)\big), \quad \forall (\mathbf{v}_h, q_h) \in W_h^k, \tag{6}$$

with

$$\mathbf{A}\big((\mathbf{u}_h,p_h),(\mathbf{v}_h,q_h)\big) \stackrel{\text{def}}{=} a_h(\mathbf{u}_h,\mathbf{v}_h) + b_h(p_h,\mathbf{v}_h) - b_h(q_h,\mathbf{u}_h) \\ + j_{\mathbf{u}}(\mathbf{u}_h,\mathbf{v}_h) + j_p(p_h,q_h), \tag{7}$$

$$a_h(\mathbf{u}_h,\mathbf{v}_h) \stackrel{\text{def}}{=} a(\mathbf{u}_h,\mathbf{v}_h) - \langle 2\nu\boldsymbol{\varepsilon}(\mathbf{u}_h)\mathbf{n},\mathbf{v}_h\rangle_{\partial\Omega} - \langle \mathbf{u}_h, 2\nu\boldsymbol{\varepsilon}(\mathbf{v}_h)\mathbf{n}\rangle_{\partial\Omega} \\ - \langle \boldsymbol{\beta}\cdot\mathbf{n}\,\mathbf{u}_h,\mathbf{v}_h\rangle_{\partial\Omega_{\text{in}}} + \left\langle \gamma_\nu \frac{\nu}{h}\mathbf{u}_h,\mathbf{v}_h\right\rangle_{\partial\Omega} + \left\langle \gamma_{\mathbf{n}} \max\left\{|\boldsymbol{\beta}|,\frac{\nu}{h}\right\}\mathbf{u}_h\cdot\mathbf{n},\mathbf{v}_h\cdot\mathbf{n}\right\rangle_{\partial\Omega}, \tag{8}$$

$$b_h(p_h,\mathbf{v}_h) \stackrel{\text{def}}{=} b(p_h,\mathbf{v}_h) + \langle p_h,\mathbf{v}_h\cdot\mathbf{n}\rangle_{\partial\Omega}, \tag{9}$$

$$j_{\mathbf{u}}(\mathbf{u}_h,\mathbf{v}_h) \stackrel{\text{def}}{=} \sum_{K\in\mathcal{T}_h} \gamma_{\boldsymbol{\beta}} \frac{h_K^2}{\|\boldsymbol{\beta}\|_{0,\infty,K}} \int_{\partial K} (\boldsymbol{\beta}_h\cdot[\![\nabla\mathbf{u}_h]\!])\cdot(\boldsymbol{\beta}_h\cdot[\![\nabla\mathbf{v}_h]\!])\,\mathrm{d}s \\ + \sum_{K\in\mathcal{T}_h} \gamma_{\text{div}} h_K^2 \|\boldsymbol{\beta}\|_{0,\infty,K} \int_{\partial K} [\![\nabla\cdot\mathbf{u}_h]\!][\![\nabla\cdot\mathbf{v}_h]\!]\,\mathrm{d}s, \tag{10}$$

$$j_p(p_h,q_h) \stackrel{\text{def}}{=} \sum_{K\in\mathcal{T}_h} \gamma_p \xi(\mathrm{Re_K}) \frac{h_K^2}{\|\boldsymbol{\beta}\|_{0,\infty,K}} \int_{\partial K} [\![\nabla p_h]\!]\cdot[\![\nabla q_h]\!]\,\mathrm{d}s, \tag{11}$$

$$\mathbf{F}\big((\mathbf{v}_h,q_h)\big) \stackrel{\text{def}}{=} (\mathbf{f},\mathbf{v}_h),$$

$\mathbf{n}$ the outward pointing normal to $\partial\Omega$, and using the notation

$$\mathrm{Re}_K \stackrel{\text{def}}{=} \frac{\|\boldsymbol{\beta}\|_{0,\infty,K}h_K}{\nu}, \quad \xi(\lambda) \stackrel{\text{def}}{=} \min\{1,\lambda\},$$
$$\langle x,y\rangle_{\partial\Omega} \stackrel{\text{def}}{=} \sum_{e\subset\partial\Omega}\int_e xy\,\mathrm{d}s, \quad \partial\Omega_{\text{in}} \stackrel{\text{def}}{=} \{\mathbf{x}\in\partial\Omega\,:\,(\boldsymbol{\beta}\cdot\mathbf{n})(\mathbf{x})<0\}.$$

The gradient jump terms serve three purposes:

1. stabilization of the convective terms (the first sum in (10));
2. giving additional control of the incompressibility condition (the second sum in (10));
3. making the discretization *inf–sup* stable (the sum in (11)).

We will see in the analysis that these three objectives are all obtained in the same fashion and that essentially the gradient jump operator can stabilize any instability provoked by a first order term. It should be noted that for the case of the incompressible Navier-Stokes equations it might be advantageous to substitute the term controlling the jumps in the divergence by the classical least squares stabilizing term $(\gamma_{div} h\nabla\cdot\mathbf{u}_h,\nabla\cdot\mathbf{v}_h)$ in order to keep down the bandwith of the matrix. However for more complex problems such as low-Mach flow the jump term is expected to have some advantages.

Assuming sufficient regularity of the exact solution the above formulation is strongly consistent.

Lemma 2.1 (Galerkin Orthogonality) *Assume that* $(\mathbf{u}, p)$*, the solution of* (1)*, belongs to the space* $[H^{3/2+\epsilon}(\Omega)]^{d+1}$*, with* $\epsilon > 0$*, and let* $(\mathbf{u}_h, p_h) \in W_h^k$ *be the solution of* (6)*. Then*

$$\mathbf{A}\big((\mathbf{u} - \mathbf{u}_h, p - p_h), (\mathbf{v}_h, q_h)\big) = 0, \quad \forall (\mathbf{v}_h, q_h) \in W_h^k.$$

Proof. This is an immediate consequence of the consistency of the standard Galerkin method and the fact that under the regularity assumptions $j_p(p, q_h) = 0$ and $j_{\mathbf{u}}(\mathbf{u}, \mathbf{v}_h) = 0$, since $[\![\nabla p]\!]_e = 0$ and $[\![\nabla \mathbf{u}]\!]_e = 0$ for all interior edges e. □

3 Stability of the method

Stability in the edge stabilization method is based on the following lemma which was proved for piecewise linear continuous approximation in [7]. Here we extend this result to arbitrary polynomial degree. Note that we give a lower bound as well. This is not needed for the analysis, but shows that in some sense the stabilizing terms are optimal.

Lemma 3.1 *There exists an interpolation operator* $\pi_h^* : [H^2(\mathcal{T}_h)]^d \longrightarrow [V_h^k]^d$ *and constants* $\gamma_{\boldsymbol{\beta}}$*,* γ_{low} *depending on the local mesh geometry and the polynomial degree, but not on the local mesh size, such that*

$$\gamma_{low} j_{\boldsymbol{\beta}}(\mathbf{v}_h, \mathbf{v}_h) \le \|h^{\frac{1}{2}}\big(\boldsymbol{\beta}_h \cdot \nabla \mathbf{v}_h - \pi_h^*(\boldsymbol{\beta}_h \cdot \nabla \mathbf{v}_h)\big)\|_{0,\Omega}^2 \le j_{\boldsymbol{\beta}}(\mathbf{v}_h, \mathbf{v}_h)$$

for all $\mathbf{v}_h \in [V_h^k]^d$*, where*

$$j_{\boldsymbol{\beta}}(\mathbf{v}_h, \mathbf{v}_h) = \gamma_{\boldsymbol{\beta}} \sum_{K \in \mathcal{T}_h} \int_{\partial K} h_K^2 |\boldsymbol{\beta}_h \cdot [\![\nabla \mathbf{v}_h]\!]|^2 \, \mathrm{d}s.$$

Proof. For each node x_i, let n_i be the number of elements containing x_i as a node. Then we define a quasi-interpolant π_h^* of degree k by

$$\pi_h^* \mathbf{v}(x_i) \stackrel{\text{def}}{=} \frac{1}{n_i} \sum_{\{K \,:\, x_i \in K\}} \mathbf{v}_{|K}(x_i), \quad \forall \mathbf{v} \in [H^2(\mathcal{T}_h)]^d.$$

For each element $K \in \mathcal{T}_h$ consider the function

$$\delta_K \stackrel{\text{def}}{=} h_K^{\frac{1}{2}}\big(\boldsymbol{\beta}_h \cdot \nabla \mathbf{v}_{h|K} - \pi_h^*(\boldsymbol{\beta}_h \cdot \nabla \mathbf{v}_h)_{|K}\big).$$

Clearly, $\delta_K(x_j) = 0$ for each interior node $x_j \in \overset{\circ}{K}$ whereas on the element edges, i.e. for all nodes $x_j \in \partial K$, we have

$$\begin{aligned}
\delta_K(x_j) &= h_K^{\frac{1}{2}} \frac{1}{n_j} \sum_{\{K' \,:\, x_j \in K'\}} \boldsymbol{\beta}_h \cdot \big(\nabla \mathbf{v}_{h|K}(x_j) - \nabla \mathbf{v}_{h|K'}(x_j)\big) \\
&= h_K^{\frac{1}{2}} \frac{1}{n_j} \sum_{\{K' \,:\, x_j \in K'\}} \sum_{e \in P(K, K')} \boldsymbol{\beta}_h(x_j) \cdot [\![\nabla \mathbf{v}_h]\!]_e(x_j), \qquad (12)
\end{aligned}$$

where $P(K, K')$ stands for the set of edges between K and K' (the shortest path). We now introduce the reference element $\hat{K}$ and, for each $K \in \mathcal{T}_h$, the affine mapping

$$F_K(\hat{\mathbf{x}}) = B_K \hat{\mathbf{x}} + \mathbf{b}_K, \quad \forall \hat{\mathbf{x}} \in \hat{K},$$

such that $F_K(\hat{K}) = K$. Finally, let φ_j^K, for $j = 1, \ldots, k$, the basis functions on K. Since $\delta_K(x_j) = 0$ for each interior node $x_j \in \overset{\circ}{K}$, $\|\delta_K \circ F_K\|_{0,\partial\hat{K}}^2 = 0$ implies that $\delta_K \circ F_K = 0$ in $\hat{K}$. Therefore, by equivalence of norms on discrete spaces, using a standard scaling argument (see [21, p. 96]) and (12), it follows that

$$\begin{aligned}
\|\delta_K\|_{0,K}^2 &= \det B_K \|\delta_K \circ F_K\|_{0,\hat{K}}^2 \\
&\leq C \det B_K \|\delta_K \circ F_K\|_{0,\partial\hat{K}}^2 \\
&= \int_{\partial\hat{K}} \frac{1}{|B_K^{-\mathrm{T}} \hat{\mathbf{n}}|} |\delta_K \circ F_K|^2 \underbrace{\det B_K |B_K^{-\mathrm{T}} \hat{\mathbf{n}}| \, \mathrm{d}\hat{s}}_{\mathrm{d}s} \\
&\leq C |B_K^{\mathrm{T}}| \int_{\partial K} |\delta_K|^2 \, \mathrm{d}s \\
&\leq C h_K \int_{\partial K} |\delta_K|^2 \, \mathrm{d}s \\
&\leq C h_K \int_{\partial K} \sum_{j=1}^{k} |\delta_K(x_j)|^2 (\varphi_j^K)^2 \, \mathrm{d}s \\
&\leq C h_K^2 \int_{\partial K} \sum_{j=1}^{k} \frac{1}{n_j} \sum_{\{K' : x_j \in K'\}} \sum_{e \in P(K,K')} |\boldsymbol{\beta}_h(x_j) \cdot [\![\nabla \mathbf{v}_h]\!]_e(x_j)|^2 (\varphi_j^K)^2 \, \mathrm{d}s \\
&\leq C h_K^2 \int_{\partial K} \sum_{j=1}^{k} \frac{1}{n_j} \sum_{e \in E(K)} |\boldsymbol{\beta}_h(x_j) \cdot [\![\nabla \mathbf{v}_h]\!]_e(x_j)|^2 (\varphi_j^K)^2 \, \mathrm{d}s \\
&\leq C h_K^2 \sum_{e \in E(K)} \int_e |\boldsymbol{\beta}_h \cdot [\![\nabla \mathbf{v}_h]\!]_e|^2 \, \mathrm{d}s,
\end{aligned}$$

where, in the two last inequalities, $E(K)$ denotes the set of edges containing some node of K. On the other hand, the local quasi regularity of $\mathcal{T}_h$ implies that the maximum number of occurrences of a edge in all the sets $E(K)$ is bounded by a fixed constant independent of h_K. Then, by summation on K, we get the upper bound

$$\begin{aligned}
\|h^{\frac{1}{2}} \big(\boldsymbol{\beta}_h \cdot \nabla \mathbf{v}_h - \pi_h^*(\boldsymbol{\beta}_h \cdot \nabla \mathbf{v}_h)\big)\|_{0,\Omega}^2 &\leq C \sum_{K \in \mathcal{T}_h} h_K^2 \sum_{e \in E(K)} \int_e |\boldsymbol{\beta}_h \cdot [\![\nabla \mathbf{v}_h]\!]_e|^2 \, \mathrm{d}s, \\
&\leq C \sum_{K \in \mathcal{T}_h} \int_{\partial K} h_K^2 |\boldsymbol{\beta}_h \cdot [\![\nabla \mathbf{v}_h]\!]|^2 \, \mathrm{d}s,
\end{aligned}$$

The lower bound follows by considering the L^2-norm of the discontinuous function δ over the reference patch $\hat{G}$ consisting of the reference element $\hat{K}$ and its nearest neighbors. Clearly if $\|\delta\|_{\hat{G}} = 0$ then $\boldsymbol{\beta}_h \cdot \nabla \mathbf{v}_h = \pi_h^* \boldsymbol{\beta}_h \cdot \nabla \mathbf{v}_h$ in $\hat{G}$. This means that $\boldsymbol{\beta}_h \cdot \nabla \mathbf{v}_h$ is continuous in $\hat{G}$ and hence $\sum_{e \in E(K)} \int_e h_K [\![\boldsymbol{\beta}_h \cdot \nabla \mathbf{v}_h]\!]^2 \, \mathrm{d}s = 0$. Hence by norm equivalence on discrete spaces we have

$$\sum_{e \in E(K)} \int_e [\![\boldsymbol{\beta}_h \cdot \nabla \mathbf{v}_h]\!]^2 \, \mathrm{d}s \le \|\delta_G\|_{0,G}^2$$

The claim then follows in the same fashion as the first part of the proof by scaling and extension to all of $\mathcal{T}_h$. $\square$

Using the same technique we immediately have the following corollary where for simplicity the lower bounds are omitted.

Corollary 3.1 *Under the same assumptions as Lemma 3.1 we have*

$$\begin{aligned} \|h^{\frac{1}{2}}\big(\nabla \cdot \mathbf{v}_h - \pi_h^*(\nabla \cdot \mathbf{v}_h)\big)\|_{0,\Omega}^2 &\le \gamma_{\mathrm{div}} \sum_{K \in \mathcal{T}_h} \int_{\partial K} h_K^2 [\![\nabla \cdot \mathbf{v}_h]\!]^2 \, \mathrm{d}s, \\ \|h^{\frac{1}{2}}\big(\nabla q_h - \pi_h^*(\nabla q_h)\big)\|_{0,\Omega}^2 &\le \gamma_p \sum_{K \in \mathcal{T}_h} \int_{\partial K} h_K^2 |[\![\nabla q_h]\!]|^2 \, \mathrm{d}s, \end{aligned} \tag{13}$$

for all $(\mathbf{v}_h, q_h) \in W_h^k$ *and with* $\gamma_{\mathrm{div}}, \gamma_p > 0$ *constants independent of* h.

Now, in order to give our main stability result, namely the inf-sup condition for $\mathbf{A}$, we define respectively the following mesh-dependent semi-norm and norm in W_h^k:

$$\begin{aligned} |\!|\!|(\mathbf{v}_h, q_h)|\!|\!|^2 &\stackrel{\text{def}}{=} \|\sigma^{\frac{1}{2}} \mathbf{v}_h\|_{0,\Omega}^2 + \|\nu^{\frac{1}{2}} \nabla \mathbf{v}_h\|_{0,\Omega}^2 + j_{\mathbf{u}}(\mathbf{v}_h, \mathbf{v}_h) + j_p(q_h, q_h) + \||\boldsymbol{\beta} \cdot \mathbf{n}|^{\frac{1}{2}} \mathbf{v}_h\|_{0,\partial\Omega_{\mathrm{in}}}^2 \\ &\quad + \|\gamma_\nu^{\frac{1}{2}} \, (\nu/h)^{\frac{1}{2}} \, \mathbf{v}_h\|_{0,\partial\Omega}^2 + \|\gamma_{\mathbf{n}}^{\frac{1}{2}} \max\left\{|\boldsymbol{\beta}|, \frac{\nu}{h}\right\}^{\frac{1}{2}} \mathbf{v}_h \cdot \mathbf{n}\|_{0,\partial\Omega}^2, \\ |\!|\!|(\mathbf{v}_h, q_h)|\!|\!|_*^2 &\stackrel{\text{def}}{=} |\!|\!|(\mathbf{v}_h, q_h)|\!|\!|^2 + \|q_h\|_{0,\Omega}^2, \end{aligned}$$

for all $(\mathbf{v}_h, q_h) \in W_h^k$.

The following lemma gives the coercivity of our operator with respect to the above mesh-dependent semi-norm.

Lemma 3.2 (Coercivity) *There exists a constant* $C > 0$*, depending only on* Ω *and* γ_ν*, such that*

$$\underbrace{a_h(\mathbf{v}_h, \mathbf{v}_h) + j_{\mathbf{u}}(\mathbf{v}_h, \mathbf{v}_h) + j_p(q_h, q_h)}_{\mathbf{A}\big((\mathbf{v}_h, q_h), (\mathbf{v}_h, q_h)\big)} \ge C |\!|\!|(\mathbf{v}_h, q_h)|\!|\!|^2,$$

for all $(\mathbf{v}_h, q_h) \in W_h^k$.

Proof. From (6) we get

$$
\begin{aligned}
&a_h(\mathbf{v}_h,\mathbf{v}_h)+j_{\mathbf{u}}(\mathbf{v}_h,\mathbf{v}_h)+j_p(q_h,q_h)\geq \|\sigma^{\frac{1}{2}}\mathbf{v}_h\|^2_{0,\Omega}\\
&+2\|\nu^{\frac{1}{2}}\varepsilon(\mathbf{v}_h)\|^2_{0,\Omega}+j_{\mathbf{u}}(\mathbf{v}_h,\mathbf{v}_h)+j_p(q_h,q_h)+\frac{1}{2}\||\boldsymbol{\beta}\cdot\mathbf{n}|^{\frac{1}{2}}\mathbf{v}_h\|^2_{0,\partial\Omega}\\
&+\|\gamma_\nu^{\frac{1}{2}}\left(\frac{\nu}{h}\right)^{\frac{1}{2}}\mathbf{v}_h\|^2_{0,\partial\Omega}+\|\gamma_{\mathbf{n}}^{\frac{1}{2}}\max\left\{|\boldsymbol{\beta}|,\frac{\nu}{h}\right\}^{\frac{1}{2}}\mathbf{v}_h\cdot\mathbf{n}\|^2_{0,\partial\Omega}-\langle 4\nu\varepsilon(\mathbf{v}_h)\mathbf{n},\mathbf{v}_h\rangle_{\partial\Omega}
\end{aligned}
\tag{14}
$$

where we used the fact that, after integration by parts,

$$
(\boldsymbol{\beta}\cdot\nabla\mathbf{v}_h,\mathbf{v}_h)=\frac{1}{2}\langle\boldsymbol{\beta}\cdot\mathbf{n}\mathbf{v}_h,\mathbf{v}_h\rangle_{\partial\Omega}\,.
$$

The last term in (14) can be bounded using the Cauchy-Schwarz inequality followed by a local inverse inequality, to obtain

$$
|\langle 4\nu\varepsilon(\mathbf{v}_h)\mathbf{n},\mathbf{v}_h\rangle_{\partial\Omega}|\leq\frac{c_I}{2\gamma_\nu}\|\nu^{\frac{1}{2}}\varepsilon(\mathbf{v}_h)\|^2_{0,\Omega}+\frac{1}{2}\|\gamma_\nu^{\frac{1}{2}}\left(\frac{\nu}{h}\right)^{\frac{1}{2}}\mathbf{v}_h\|^2_{0,\partial\Omega}.
$$

We choose $\gamma_\nu>\frac{c_I}{2}>0$ and we conclude the proof using Korn's inequality. $\Box$

We now state the stability result for the discrete formulation (6) the proof of which can be found in Appendix A.

Theorem 3.1 (Stability) *There exists a constant $C>0$, independent of h and ν, such that*

$$
\sup_{\mathbf{0}\neq(\mathbf{v}_h,q_h)\in W_h^k}\frac{\mathbf{A}\big((\mathbf{u}_h,p_h),(\mathbf{v}_h,q_h)\big)}{|\!|\!|(\mathbf{v}_h,q_h)|\!|\!|_*}\geq C|\!|\!|(\mathbf{u}_h,p_h)|\!|\!|_*,
$$

for all $(\mathbf{u}_h,p_h)\in W_h^k$.

Remark 3.1 *The previous theorem implies that the discrete problem* (6) *admits a unique solution.*

4 Convergence of the method

Note that the stabilized formulation is not entirely independent of the local Reynolds number. The parameter for the pressure stabilization scales as $h^2/|\boldsymbol{\beta}|$ when the local Reynolds number Re_K is big and as h^3/ν when Re_K is small. We will now show that this scaling gives optimal a priori error estimates in the high (local) Reynolds number regime when the solution is smooth, $(\mathbf{u},p)\in[H^{k+1}(\Omega)]^{d+1}$ and in the low (local) Reynolds number regime under standard regularity assumptions. We then prove, using the Aubin-Nitsche duality technique (see, e.g. [17]), that the velocities have optimal convergence order also in the L^2-norm when the local Reynolds number is low, without any additional modifications of the stabilization. First of all we prove the following property of the L^2-projection on the space V_h^k, we define a triple norm and show an approximability result.

Lemma 4.1 *Let $\mathcal{T}_h$ be a locally quasi uniform mesh and $l \geq 1$ then*

$$\sum_{K \in \mathcal{T}_h} \|h^l \nabla \pi_h u\|_{l,\Omega}^2 \leq C\|u\|_{1,\Omega}^2,$$

with $C > 0$ a constant independent of h.

Proof. The proof is an immediate consequence of the H^1-stability of the L^2-projection on locally quasi-uniform meshes (see [4]) in combination with a local inverse estimate, namely,

$$\sum_{K \in \mathcal{T}_h} \|h_K^l \nabla \pi_h u\|_{l,K}^2 \leq C\|\nabla \pi_h u\|_{0,\Omega}^2 \leq C\|u\|_{1,\Omega}^2.$$

□

Lemma 4.2 *Let $(\mathbf{u}, p) \in [H^{k+1}(\Omega)]^{d+1}$, for some $k \geq 1$. Under the assumption of local quasi-uniformity for the mesh we have*

$$\begin{aligned}|||(\mathbf{u} - \pi_h \mathbf{u}, p - \pi_h p)||| \leq C \left(\sigma^{\frac{1}{2}} h^{k+1} + \max\left\{\nu, \|\boldsymbol{\beta}\|_{0,\infty,\Omega} h\right\}^{\frac{1}{2}} h^k\right) \|\mathbf{u}\|_{k+1,\Omega} \\ + C \max_{K \in \mathcal{T}_h} \|\boldsymbol{\beta}\|_{0,\infty,K}^{-\frac{1}{2}} h^{k+\frac{1}{2}} \|p\|_{k+1,\Omega},\end{aligned}$$

with $C > 0$ a constant depending only on γ_ν ,$\gamma_{\mathbf{n}}$, $\gamma_{\boldsymbol{\beta}}$, γ_{div} and γ_p.

Proof. By the standard error estimate for the L^2-projection (see [17]) we have

$$\|\sigma^{\frac{1}{2}}(\mathbf{u} - \pi_h \mathbf{u})\|_{0,\Omega} \leq C\sigma^{\frac{1}{2}} h^{k+1} \|\mathbf{u}\|_{k+1,\Omega}.$$

Using now the H^1-stability of the L^2-projection on locally quasi-uniform meshes (see [4]) we get, with i_h denoting the nodal interpolant and a standard interpolation error estimate,

$$\|\nu^{\frac{1}{2}} \nabla(i_h \mathbf{u} - \pi_h \mathbf{u})\|_{0,\Omega} \leq C\|\nu^{\frac{1}{2}} \nabla(i_h \mathbf{u} - \mathbf{u})\|_{0,\Omega} \leq C\nu^{\frac{1}{2}} h^k \|\mathbf{u}\|_{k+1,\Omega}.$$

We treat the boundary terms using the trace inequality (4) in combination with the above estimates for the L^2-norm and the H^1-semi norm, e.g.,

$$\begin{aligned}\|\mathbf{u} - \pi_h \mathbf{u}\|_{0,\partial\Omega}^2 &\leq C \sum_{e \subset \partial\Omega} \left(h_e^{-1} \|\mathbf{u} - \pi_h \mathbf{u}\|_{0,K_e}^2 + h_e \|\mathbf{u} - \pi_h \mathbf{u}\|_{1,K_e}^2\right) \\ &\leq C h^{2k+1} \|\mathbf{u}\|_{k+1,\Omega}^2,\end{aligned} \tag{15}$$

where K_e denotes the simplex such that $e \subset \partial K_e \cap \partial\Omega$. The interior penalty terms are treated in the same fashion as the boundary terms. We have

$$\begin{aligned}j_{\mathbf{u}}(\mathbf{u} - \pi_h \mathbf{u}, \mathbf{u} - \pi_h \mathbf{u}) \leq j_{\mathbf{u}}(i_h \mathbf{u} - \pi_h \mathbf{u}, i_h \mathbf{u} - \pi_h \mathbf{u}) \\ + j_{\mathbf{u}}(\mathbf{u} - i_h \mathbf{u}, \mathbf{u} - i_h \mathbf{u}).\end{aligned} \tag{16}$$

The first term in this inequality can be estimated using the trace inequality (4) and then Lemma 4.1, yielding

$$\begin{aligned} j_{\mathbf{u}}(i_h\mathbf{u}-\pi_h\mathbf{u}, i_h\mathbf{u}-\pi_h\mathbf{u}) &= \sum_{K\in\mathcal{T}_h}\Big[\gamma_{\boldsymbol\beta}\frac{h_K^2}{\|\boldsymbol\beta\|_{0,\infty,K}}\int_{\partial K}|\boldsymbol\beta_h\cdot[\![\nabla(i_h\mathbf{u}-\pi_h\mathbf{u})]\!]|^2\,\mathrm{d}s \\ &+ \gamma_{\mathrm{div}}h_K^2\|\boldsymbol\beta\|_{0,\infty,K}\int_{\partial K}[\![\nabla\cdot(i_h\mathbf{u}-\pi_h\mathbf{u})]\!]^2\,\mathrm{d}s\Big] \\ &\le C\|\boldsymbol\beta\|_{0,\infty,\Omega}\sum_{K\in\mathcal{T}_h}h_K^2\|\nabla(i_h\mathbf{u}-\pi_h\mathbf{u})\|_{0,\partial K}^2 \\ &\le C\|\boldsymbol\beta\|_{0,\infty,\Omega}\sum_{K\in\mathcal{T}_h}\left(h_K\|\nabla(i_h\mathbf{u}-\pi_h\mathbf{u})\|_{0,K}^2+h_K^3\|\nabla(i_h\mathbf{u}-\pi_h\mathbf{u})\|_{1,K}^2\right) \\ &\le C\|\boldsymbol\beta\|_{0,\infty,\Omega}h^{2k+1}\|\mathbf{u}\|_{k+1,\Omega}^2, \end{aligned} \tag{17}$$

For the second term in (16) we obtain, in the same fashion,

$$j_{\mathbf{u}}(\mathbf{u}-i_h\mathbf{u},\mathbf{u}-i_h\mathbf{u}) \le C\|\boldsymbol\beta\|_{0,\infty,\Omega}h^{2k+1}\|\mathbf{u}\|_{k+1,\Omega}^2. \tag{18}$$

Finally, the pressure jump term is treated using the same argument and the fact that $\xi(\mathrm{Re}_K)\le 1$. This yields,

$$j_p(p-\pi_h p, p-\pi_h p) \le C\max_{K\in\mathcal{T}_h}\|\boldsymbol\beta\|_{0,\infty,K}^{-1}h^{2k+1}\|p\|_{k+1,\Omega}^2,$$

which completes the proof. □

4.1 Energy norm error estimate, smooth solutions

In this section we prove convergence in the triple norm. These results are optimal independently of the local Reynolds number when the exact solution is sufficiently smooth.

Theorem 4.1 *Assume that* $(\mathbf{u},p)\in[H^{k+1}(\Omega)]^{d+1}$, *for some* $k\ge 1$, *is the solution of* (1) *and* $(\mathbf{u}_h,p_h)\in W_h^k$ *be the solution of* (6). *Then*

$$\begin{aligned} |\!|\!|(\mathbf{u}-\mathbf{u}_h,p-p_h)|\!|\!| &\le C\left[\left(\sigma^{\frac12}+\frac{\|\boldsymbol\beta\|_{1,\infty,\Omega}}{\sigma^{1/2}}\right)h^{k+1}+\max\{\nu,\|\boldsymbol\beta\|_{0,\infty,\Omega}h\}^{\frac12}h^k\right]\|\mathbf{u}\|_{k+1,\Omega} \\ &+ C\max_{K\in\mathcal{T}_h}\|\boldsymbol\beta\|_{0,\infty,K}^{-\frac12}h^{k+\frac12}\|p\|_{k+1,\Omega}, \end{aligned}$$

with $C>0$ *a constant depending only on* γ_ν ,$\gamma_{\mathbf{n}}$, $\gamma_{\boldsymbol\beta}$, γ_{div} *and* γ_p.

Proof. Let decompose the error $(\mathbf{u}-\mathbf{u}_h,p-p_h)$ in two parts

$$\begin{aligned} \mathbf{u}-\mathbf{u}_h &= \underbrace{\mathbf{u}-\pi_h\mathbf{u}}_{\mathbf{e}^\pi}+\underbrace{\pi_h\mathbf{u}-\mathbf{u}_h}_{-\mathbf{e}_h} = \mathbf{e}^\pi-\mathbf{e}_h, \\ p-p_h &= \underbrace{p-\pi_h p}_{y^\pi}+\underbrace{\pi_h p-p_h}_{-y_h} = y^\pi-y_h. \end{aligned}$$

It follows then that

$$|||(\mathbf{u}-\mathbf{u}_h, p-p_h)||| \leq |||(\mathbf{e}^\pi, y^\pi)||| + |||(\mathbf{e}_h, y_h)|||.$$

Lemma 4.2 gives an estimate for $|||(\mathbf{e}^\pi, y^\pi)|||$, hence it suffices to study $|||(\mathbf{e}_h, y_h)|||$. Using coercivity and orthogonality, Lemmas 3.2 and 2.1, we get

$$\begin{aligned} C|||(\mathbf{e}_h, y_h)|||^2 &\leq \mathbf{A}\big((\mathbf{e}_h, y_h), (\mathbf{e}_h, y_h)\big) \\ &= \mathbf{A}\big((\mathbf{e}^\pi, y^\pi), (\mathbf{e}_h, y_h)\big) \\ &= a_h(\mathbf{e}^\pi, \mathbf{e}_h) + b_h(y^\pi, \mathbf{e}_h) - b_h(y_h, \mathbf{e}^\pi) \\ &\quad + j_{\mathbf{u}}(\mathbf{e}^\pi, \mathbf{e}_h) + j_p(y^\pi, y_h) \end{aligned} \tag{19}$$

By an application of the Cauchy-Schwarz inequality in the symmetric part of the discrete elliptic operator and integrating by parts in the convective term we obtain

$$\begin{aligned} a_h(\mathbf{e}^\pi, \mathbf{e}_h) \leq |||(\mathbf{e}^\pi, 0)||| \, |||(\mathbf{e}_h, 0)||| + |(\mathbf{e}^\pi, \boldsymbol{\beta}\cdot\nabla\mathbf{e}_h)| \\ - \langle 2\nu\boldsymbol{\varepsilon}(\mathbf{e}^\pi)\mathbf{n}, \mathbf{e}_h\rangle_{\partial\Omega} - \langle \mathbf{e}^\pi, 2\nu\boldsymbol{\varepsilon}(\mathbf{e}_h)\mathbf{n}\rangle_{\partial\Omega}\,, \end{aligned}$$

where, for simplicity, the boundary term from the integration by parts has been included in the first term on the right hand side. We note that in the same way we have, using the Cauchy-Schwarz inequality, a trace inequality and a local inverse inequality,

$$\langle 2\nu\boldsymbol{\varepsilon}(\mathbf{e}_h)\mathbf{n}, \mathbf{e}^\pi\rangle_{\partial\Omega} \leq C|||(\mathbf{e}_h, 0)||| \, |||(\mathbf{e}^\pi, 0)|||.$$

For the second boundary term we use the Cauchy-Schwarz inequality followed by a trace inequality and an approximation argument, similar to (16)-(17), to obtain

$$\langle 2\nu\boldsymbol{\varepsilon}(\mathbf{e}^\pi)\mathbf{n}, \mathbf{e}_h\rangle_{\partial\Omega} \leq C\nu^{\frac{1}{2}}h^k\|\mathbf{u}\|_{k+1,\Omega}|||(\mathbf{e}_h, 0)|||.$$

The convective term is controlled using a local inverse inequality and the orthogonality of the L^2-projection, after having replaced the continuous velocity field $\boldsymbol{\beta}$ by its piecewise linear interpolant $\boldsymbol{\beta}_h$,

$$\begin{aligned} |(\mathbf{e}^\pi, \boldsymbol{\beta}\cdot\nabla\mathbf{e}_h)| &\leq |(\mathbf{e}^\pi, (\boldsymbol{\beta}-\boldsymbol{\beta}_h)\cdot\nabla\mathbf{e}_h)| + |(\mathbf{e}^\pi, \boldsymbol{\beta}_h\cdot\nabla\mathbf{e}_h)| \\ &\leq \sum_{K\in\mathcal{T}_h} \|\boldsymbol{\beta}\|_{1,\infty,K}\|\mathbf{e}^\pi\|_{0,K}h_K\|\nabla\mathbf{e}_h\|_{0,K} \\ &\quad + |(\mathbf{e}^\pi, \boldsymbol{\beta}_h\cdot\nabla\mathbf{e}_h - \pi_h^*(\boldsymbol{\beta}_h\cdot\nabla\mathbf{e}_h))| \\ &\leq C\frac{\|\boldsymbol{\beta}\|_{1,\infty,\Omega}}{\sigma^{1/2}}h^{k+1}\|\mathbf{u}\|_{k+1,\Omega}|||(\mathbf{e}_h, 0)||| \\ &\quad + \|h^{-\frac{1}{2}}\mathbf{e}^\pi\|_{0,\Omega}\|h^{\frac{1}{2}}(\boldsymbol{\beta}_h\cdot\nabla\mathbf{e}_h - \pi_h^*(\boldsymbol{\beta}_h\cdot\nabla\mathbf{e}_h))\|_{0,\Omega}. \end{aligned}$$

Now we apply Lemma 3.1 to obtain

$$\begin{aligned} \|h^{-\frac{1}{2}}\mathbf{e}^\pi\|_{0,\Omega}\|h^{\frac{1}{2}}(\boldsymbol{\beta}_h\cdot\nabla\mathbf{e}_h - \pi_h^*\boldsymbol{\beta}_h\cdot\nabla\mathbf{e}_h)\|_{0,\Omega} &\leq \|h^{-\frac{1}{2}}\mathbf{e}^\pi\|_{0,\Omega}\|\boldsymbol{\beta}\|_{0,\infty,\Omega}^{\frac{1}{2}}j_{\mathbf{u}}(\mathbf{e}_h, \mathbf{e}_h)^{\frac{1}{2}} \\ &\leq C\|\boldsymbol{\beta}\|_{0,\infty,\Omega}^{\frac{1}{2}}h^{k+\frac{1}{2}|}\|\mathbf{u}\|_{k+1,\Omega}|||(\mathbf{e}_h, 0)|||. \end{aligned}$$

Collecting terms we have

$$
\begin{aligned}
a_h(\mathbf{e}^\pi, \mathbf{e}_h) \leq\, & C|\!|\!|(\mathbf{e}^\pi, 0)|\!|\!|\,|\!|\!|(\mathbf{e}_h, 0)|\!|\!| \\
& + C\nu^{\frac{1}{2}} h^k \|\mathbf{u}\|_{k+1,\Omega}\, |\!|\!|(\mathbf{e}_h, 0)|\!|\!| \\
& + C\frac{\|\boldsymbol{\beta}\|_{1,\infty,\Omega}}{\sigma^{1/2}} h^{k+1} \|\mathbf{u}\|_{k+1,\Omega} |\!|\!|(\mathbf{e}_h, 0)|\!|\!| \\
& + C\|\boldsymbol{\beta}\|^{\frac{1}{2}}_{0,\infty,\Omega} h^{k+\frac{1}{2}} \|\mathbf{u}\|_{k+1,\Omega} |\!|\!|(\mathbf{e}_h, 0)|\!|\!|.
\end{aligned}
\tag{20}
$$

For the second term we have, using the orthogonality of the L^2-projection and replacing $\mathbf{u}$ with p in (15),

$$
\begin{aligned}
b_h(y^\pi, \mathbf{e}_h) &= -(y^\pi, \nabla\cdot\mathbf{e}_h - \pi_h^*(\nabla\cdot\mathbf{e}_h)) + \langle y^\pi, \mathbf{e}_h\cdot\mathbf{n}\rangle_{\partial\Omega} \\
&\leq \|h^{-\frac{1}{2}} y^\pi\|_{0,\Omega} \|h^{\frac{1}{2}}(\nabla\cdot\mathbf{e}_h - \pi_h^*(\nabla\cdot\mathbf{e}_h))\|_{0,\Omega} \\
&\quad + C \max_{K\in\mathcal{T}_h} \left\{ \min\left\{ \|\boldsymbol{\beta}\|^{-1}_{0,\infty,K}, \frac{h_K}{\nu} \right\} \right\}^{\frac{1}{2}} \|y^\pi\|_{0,\partial\Omega} |\!|\!|(\mathbf{e}_h, 0)|\!|\!| \\
&\leq C \max_{K\in\mathcal{T}_h} \|\boldsymbol{\beta}\|^{-\frac{1}{2}}_{0,\infty,K} \|h^{-\frac{1}{2}} y^\pi\|_{0,\Omega} j_{\mathbf{u}}(\mathbf{e}_h, \mathbf{e}_h)^{\frac{1}{2}} \\
&\quad + C \max_{K\in\mathcal{T}_h} \left\{ \min\left\{ \|\boldsymbol{\beta}\|^{-1}_{0,\infty,K}, \frac{h_K}{\nu} \right\} \right\}^{\frac{1}{2}} h^{k+\frac{1}{2}} \|p\|_{k+1,\Omega} |\!|\!|(\mathbf{e}_h, 0)|\!|\!| \\
&\leq C \max_{K\in\mathcal{T}_h} \|\boldsymbol{\beta}\|^{-\frac{1}{2}}_{0,\infty,K} h^{k+\frac{1}{2}} \|p\|_{k+1,\Omega} |\!|\!|(\mathbf{e}_h, 0)|\!|\!|.
\end{aligned}
\tag{21}
$$

In a similar fashion, after integration by parts in the third term, one obtains

$$
\begin{aligned}
b_h(y_h, \mathbf{e}^\pi) &= -(y_h, \nabla\cdot\mathbf{e}^\pi) + \langle y_h, \mathbf{e}^\pi\cdot\mathbf{n}\rangle_{\partial\Omega} = (\nabla y_h, \mathbf{e}^\pi) \\
&= (\nabla y_h - \pi_h^*(\nabla y_h), \mathbf{e}^\pi) \\
&\leq \|h^{\frac{1}{2}}(\nabla y_h - \pi_h^*(\nabla y_h))\|_{0,\Omega} \|h^{-\frac{1}{2}}\mathbf{e}^\pi\|_{0,\Omega} \\
&\leq Ch^{-\frac{1}{2}} \max_{K\in\mathcal{T}_h} \{\nu, \|\boldsymbol{\beta}\|_{0,\infty,K} h_K\}^{\frac{1}{2}} j_p(y_h, y_h)^{\frac{1}{2}} h^{k+\frac{1}{2}} \|\mathbf{u}\|_{k+1,\Omega} \\
&\leq C \max\{\nu, \|\boldsymbol{\beta}\|_{0,\infty,\Omega} h\}^{\frac{1}{2}} |\!|\!|(0, y_h)|\!|\!| h^k \|\mathbf{u}\|_{k+1,\Omega}.
\end{aligned}
\tag{22}
$$

Finally, the interior penalty terms satisfy

$$
j_{\mathbf{u}}(\mathbf{e}^\pi, \mathbf{e}_h) + j_p(y^\pi, y_h) \leq C|\!|\!|(\mathbf{e}^\pi, y^\pi)|\!|\!|\,|\!|\!|(\mathbf{e}_h, y_h)|\!|\!|. \tag{23}
$$

We conclude the proof collecting the results of (20)-(23) in (19) and applying the approximation Lemma 4.2. $\square$

4.1.1 Recovering the pressure

In the a priori error estimate provided by Theorem 4.1 the pressure error is not controlled in a norm. In this section, we show how to recover control of the L^2-norm of the pressure error.

Theorem 4.2 *Let $(\mathbf{u}, p) \in [H^{k+1}(\Omega)]^{d+1}$, for some $k \geq 1$, be solution of (1) and $(\mathbf{u}_h, p_h)$ be the solution of (6), then there holds*

$$\|p - p_h\|_{0,\Omega} \leq C \left(h + \max\{\nu, \|\boldsymbol{\beta}\|_{0,\infty,\Omega} h\}^{\frac{1}{2}} \right) h^k \big(\|p\|_{k+1,\Omega} + \|\mathbf{u}\|_{k+1,\Omega} \big),$$

with $C > 0$ constant independent of h and ν.

Proof. Following [21, Corollary 2.4], there exists $\mathbf{v}_p \in [H_0^1(\Omega)]^d$ such that

$$\nabla \cdot \mathbf{v}_p = p - p_h, \quad \|\mathbf{v}_p\|_{1,\Omega} \leq C \|p - p_h\|_{0,\Omega}. \tag{24}$$

Using this in combination with Galerkin orthogonality we readily obtain

$$\begin{aligned} \|p - p_h\|_{0,\Omega}^2 &= (p - p_h, \nabla \cdot \mathbf{v}_p) \\ &= (p - p_h, \nabla \cdot (\mathbf{v}_p - \pi_h \mathbf{v}_p)) + \langle p - p_h, \pi_h \mathbf{v}_p \cdot \mathbf{n} \rangle_{\partial\Omega} \\ &\quad + a_h(\mathbf{u} - \mathbf{u}_h, \pi_h \mathbf{v}_p) + j_{\mathbf{u}}(\mathbf{u} - \mathbf{u}_h, \pi_h \mathbf{v}_p). \end{aligned}$$

Thus, integrating by parts, it follows that

$$\begin{aligned} \|p - p_h\|_{0,\Omega}^2 = &\underbrace{(\nabla(p - p_h), \mathbf{v}_p - \pi_h \mathbf{v}_p)}_{\mathrm{I}} + \underbrace{a_h(\mathbf{u} - \mathbf{u}_h, \pi_h \mathbf{v}_p)}_{\mathrm{II}} \\ &+ \underbrace{j_{\mathbf{u}}(\mathbf{u} - \mathbf{u}_h, \pi_h \mathbf{v}_p)}_{\mathrm{III}}. \end{aligned} \tag{25}$$

For the first term, using the orthogonality of the L^2-projection, Cauchy-Schwarz inequality, Corollary 3.1, (24) and approximation, we get

$$\begin{aligned} \mathrm{I} &= (\nabla(p - p_h) - \pi_h \nabla p + \pi_h^* \nabla p_h, \mathbf{v}_p - \pi_h \mathbf{v}_p) \\ &\leq \big(\|h(\nabla p - \pi_h \nabla p)\|_{0,\Omega} + \|h(\nabla p_h - \pi_h^* \nabla p_h)\|_{0,\Omega} \big) \|h^{-1}(\mathbf{v}_p - \pi_h \mathbf{v}_p)\|_{0,\Omega} \\ &\leq C \left(h^{k+1} \|p\|_{k+1,\Omega} + \max\{\nu, \|\boldsymbol{\beta}\|_{0,\infty,\Omega} h\}^{\frac{1}{2}} j_p(p_h, p_h)^{\frac{1}{2}} \right) \|p - p_h\|_{0,\Omega}. \end{aligned}$$

Consider now the second and third terms in (25). Using the definition of the bilinear form a_h (8), we have, after integration by parts in the convective term,

$$\begin{aligned} \mathrm{II} + \mathrm{III} \leq\ &|\!|\!|(\mathbf{u} - \mathbf{u}_h, 0)|\!|\!| \, |\!|\!|(\pi_h \mathbf{v}_p, 0)|\!|\!| + (\mathbf{u} - \mathbf{u}_h, \boldsymbol{\beta} \cdot \nabla \pi_h \mathbf{v}_p) \\ &- \langle 2\nu \boldsymbol{\varepsilon}(\mathbf{u} - \mathbf{u}_h)\mathbf{n}, \pi_h \mathbf{v}_p \rangle_{\partial\Omega} - \langle \mathbf{u} - \mathbf{u}_h, 2\nu \boldsymbol{\varepsilon}(\pi_h \mathbf{v}_p)\mathbf{n} \rangle_{\partial\Omega}. \end{aligned} \tag{26}$$

For the convective term we have, using the H^1 stability of the L^2-projection and (24),

$$\begin{aligned} (\mathbf{u} - \mathbf{u}_h, \boldsymbol{\beta} \cdot \nabla \pi_h \mathbf{v}_p) &\leq \|\boldsymbol{\beta}\|_{0,\infty,\Omega} \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} \|\nabla \pi_h \mathbf{v}_p\|_{0,\Omega} \\ &\leq C \|\boldsymbol{\beta}\|_{0,\infty,\Omega} \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} \|p - p_h\|_{0,\Omega}. \end{aligned}$$

The boundary terms are controlled in the following fashion

$$\begin{aligned}\langle 2\nu\varepsilon(\mathbf{u}-\mathbf{u}_h)\mathbf{n},\pi_h\mathbf{v}_p\rangle_{\partial\Omega}+\langle \mathbf{u}-\mathbf{u}_h,2\nu\varepsilon(\pi_h\mathbf{v}_p)\mathbf{n}\rangle_{\partial\Omega}\\ \leq \|(2\nu h)^{\frac{1}{2}}\varepsilon(\mathbf{u}-\mathbf{u}_h)\|_{0,\partial\Omega}\|(2\nu)^{\frac{1}{2}}h^{-\frac{1}{2}}\pi_h\mathbf{v}_p\|_{0,\partial\Omega}\\ +\|(2\nu h)^{\frac{1}{2}}\varepsilon(\pi_h\mathbf{v}_p)\|_{0,\partial\Omega}\|(2\nu)^{\frac{1}{2}}h^{-\frac{1}{2}}(\mathbf{u}-\mathbf{u}_h)\|_{0,\partial\Omega}.\end{aligned}$$

In addition, we have

$$\begin{aligned}\|(2\nu h)^{\frac{1}{2}}\varepsilon(\mathbf{u}-\mathbf{u}_h)\|_{0,\partial\Omega}\leq \|(2\nu h)^{\frac{1}{2}}\varepsilon(\mathbf{u}-\pi_h\mathbf{u})\|_{0,\partial\Omega}\\ +\|(2\nu h)^{\frac{1}{2}}\varepsilon(\pi_h\mathbf{u}-\mathbf{u}_h)\|_{0,\partial\Omega},\end{aligned}$$

where the first term satisfies, using an elementwise trace inequality followed by approximation,

$$\|(2\nu h)^{\frac{1}{2}}\varepsilon(\mathbf{u}-\pi_h\mathbf{u})\|_{0,\partial\Omega}\leq C\nu^{\frac{1}{2}}h^k\|\mathbf{u}\|_{k+1,\Omega},$$

and the second, using a trace inequality followed by a local inverse inequality,

$$\|(2\nu h)^{\frac{1}{2}}\varepsilon(\pi_h\mathbf{u}-\mathbf{u}_h)\|_{0,\partial\Omega}\leq C|\!|\!|(\pi_h\mathbf{u}-\mathbf{u}_h,0)|\!|\!|.$$

In the same fashion we conclude that

$$\|(2\nu)^{\frac{1}{2}}h^{-\frac{1}{2}}\pi_h\mathbf{v}_p\|_{0,\partial\Omega}\leq C|\!|\!|(\pi_h\mathbf{v}_p,0)|\!|\!|,$$

and

$$\|(2\nu h)^{\frac{1}{2}}\varepsilon(\pi_h\mathbf{v}_p)\|_{0,\partial\Omega}\leq C|\!|\!|(\pi_h\mathbf{v}_p,0)|\!|\!|.$$

We end the proof by noting that $|\!|\!|(\pi_h\mathbf{v}_p,0)|\!|\!|\leq C\|p-p_h\|_{0,\Omega}$. $\square$

Remark 4.1 *A similar argument is used in Appendix A to prove Theorem 3.1.*

4.2 Energy norm estimate for $\mathbf{u}\in[H^2(\Omega)]^d$, $p\in H^1(\Omega)$

In the case we do not have the ideal regularity assumed in the previous theorem we still get optimal order estimates in the low local Reynolds number regime. The main difference from the point of view of analysis is that the pressure stabilization term has to be treated as an inconsistent term. With this aim the following lemma will prove useful

Lemma 4.3 *Let $p\in H^1(\Omega)$ then*

$$j_p(\pi_h p,\pi_h p)\leq C\max_{K\in\mathcal{T}_h}\left\{\min\left\{\|\boldsymbol{\beta}\|^{-1}_{0,\infty,K},\frac{h_K}{\nu}\right\}\right\}h\|p\|^2_{1,\Omega}.$$

Proof. The proof follows by the trace inequality an inverse inequality and Lemma 4.1. $\square$

Clearly when the local Reynolds number is low we have $j_p(\pi_h p, \pi_h p) \leq Ch^2 \|p\|^2_{1,\Omega}$. Since p no longer has the regularity required for $j_p(p,p)$ to make sense the triple norm must be redefined. We simply use a triple norm without the pressure penalty term, namely,

$$\|\!|\mathbf{v}_h|\!\|^2_{\mathbf{u}} = \|\sigma^{\frac{1}{2}} \mathbf{v}_h\|^2_{0,\Omega} + \|\nu^{\frac{1}{2}} \nabla \mathbf{v}_h\|^2_{0,\Omega} + j_{\mathbf{v}}(\mathbf{v}_h, \mathbf{v}_h) + \||\boldsymbol{\beta} \cdot \mathbf{n}|^{\frac{1}{2}} \mathbf{v}_h\|^2_{0,\partial\Omega_{\text{in}}}$$
$$+ \|(\gamma_\nu \nu)^{\frac{1}{2}} h^{-\frac{1}{2}} \mathbf{v}_h\|^2_{0,\partial\Omega} + \|\gamma_{\mathbf{n}}^{\frac{1}{2}} \max\left\{|\boldsymbol{\beta}|, \frac{\nu}{h}\right\}^{\frac{1}{2}} \mathbf{v}_h \cdot \mathbf{n}\|^2_{0,\partial\Omega},$$

for all $\mathbf{v}_h \in [V_h^k]^d$.

Lemma 4.4 (Approximability) *Let* $\mathbf{u} \in [H^2(\Omega)]^d$ *then there holds*

$$\|\!|\mathbf{u} - \pi_h \mathbf{u}|\!\|_{\mathbf{u}} \leq C \left(\sigma^{\frac{1}{2}} h^2 + \max\left\{\nu, \|\boldsymbol{\beta}\|_{0,\infty,\Omega} h\right\}^{\frac{1}{2}} h \right) \|\mathbf{u}\|_{2,\Omega},$$

Proof. Identical to the proof of Lemma 4.2. $\square$

Theorem 4.3 *Let* $(\mathbf{u}, p) \in [H^2(\Omega)]^d \times H^1(\Omega)$ *be the solution of* (1) *and* $(\mathbf{u}_h, p_h) \in W_h^k$ *be the solution of* (6)*, then*

$$\|\!|\mathbf{u} - \mathbf{u}_h|\!\|_{\mathbf{u}} \leq C \left[\left(\sigma^{\frac{1}{2}} + \frac{\|\boldsymbol{\beta}\|_{1,\infty,\Omega}}{\sigma^{1/2}} \right) h^2 + \max\{\nu, \|\boldsymbol{\beta}\|_{0,\infty,\Omega} h\}^{\frac{1}{2}} h \right] \|\mathbf{u}\|_{2,\Omega}$$
$$+ C \max_{K \in \mathcal{T}_h} \left\{ \min\left\{ \|\boldsymbol{\beta}\|^{-1}_{0,\infty,K}, \frac{h_K}{\nu} \right\} \right\}^{\frac{1}{2}} h^{\frac{1}{2}} \|p\|_{1,\Omega}.$$

Proof. We will only outline the proof here giving special focus on how to treat the non-consistent interior penalty term for the pressure. As in the previous case it is sufficient to prove the convergence of $\mathbf{e}_h = \mathbf{u}_h - \pi_h \mathbf{u}$.

Using coercivity (Lemma 3.2) as previously and recalling that

$$\|\!|(\mathbf{e}_h, y_h)|\!\|^2 = \|\!|\mathbf{e}_h|\!\|^2_{\mathbf{u}} + j_p(y_h, y_h),$$

where $y_h = p_h - \pi_h p$, we have

$$C\|\!|(\mathbf{e}_h, y_h)|\!\|^2 \leq a_h(\mathbf{e}_h, \mathbf{e}_h) + j_{\mathbf{u}}(\mathbf{e}_h, \mathbf{e}_h) + j_p(y_h, y_h). \tag{27}$$

The modified Galerkin orthogonality takes the form

$$a_h(\mathbf{u}_h - \mathbf{u}, \mathbf{v}_h) + b_h(p_h - p, \mathbf{v}_h) + j_{\mathbf{u}}(\mathbf{u}_h - \mathbf{u}, \mathbf{v}_h)$$
$$- b_h(q_h, \mathbf{u}_h - \mathbf{u}) = -j_p(p_h, q_h), \tag{28}$$

for all $(\mathbf{v}_h, q_h) \in W_h^k$. Using now (28) in (27) we have

$$\begin{aligned} C\|\!|(\mathbf{e}_h, y_h)|\!\|^2 \leq\ & a_h(\mathbf{e}^\pi, \mathbf{e}_h) - b_h(y^\pi, \mathbf{e}_h) + b_h(y_h, \mathbf{e}^\pi) + j_{\mathbf{u}}(\mathbf{e}^\pi, \mathbf{e}_h) \\ & - j_p(\pi_h p, y_h). \end{aligned} \tag{29}$$

The first four terms in the right hand side of (29) are treated as in the proof of Theorem 4.1. This yields

$$\begin{aligned}
a_h(\mathbf{e}^\pi, \mathbf{e}_h) &\leq C\Bigg(|||(\mathbf{e}^\pi, 0)||| + \nu^{\frac{1}{2}} h \|\mathbf{u}\|_{2,\Omega} + \frac{\|\boldsymbol{\beta}\|_{1,\infty,\Omega}}{\sigma^{1/2}} h^2 \|\mathbf{u}\|_{2,\Omega} \\
&\qquad + \|\boldsymbol{\beta}\|^{\frac{1}{2}}_{0,\infty,\Omega} h^{\frac{3}{2}} \|\mathbf{u}\|_{2,\Omega} \Bigg) |||(\mathbf{e}_h, y_h)|||, \\
b_h(y_h, \mathbf{e}^\pi) &\leq C \max\{\nu, \|\boldsymbol{\beta}\|_{0,\infty,\Omega} h\}^{\frac{1}{2}} h \|\mathbf{u}\|_{2,\Omega} |||(\mathbf{e}_h, y_h)|||, \\
j_{\mathbf{u}}(\mathbf{e}^\pi, \mathbf{e}_h) &\leq \|\mathbf{e}^\pi\|_{\mathbf{u}} |||(\mathbf{e}_h, y_h)|||.
\end{aligned}$$

For the second therm in (29) we first recall that

$$\begin{aligned}
(y^\pi, \nabla \cdot \mathbf{e}_h) &\leq \nu^{-\frac{1}{2}} \|y^\pi\|_{0,\Omega} |||(\mathbf{e}_h, 0)||| \\
&\leq C \frac{h}{\nu} \|p\|_{1,\Omega} |||(\mathbf{e}_h, 0)|||,
\end{aligned}$$

and hence, by combining this estimation with the one using the technique of (21) we obtain

$$b_h(y^\pi, \mathbf{e}_h) \leq C \max_{K \in \mathcal{T}_h} \left\{ \min\left\{ \|\boldsymbol{\beta}\|^{-1}_{0,\infty,K}, \frac{h_K}{\nu} \right\} \right\}^{\frac{1}{2}} h^{\frac{1}{2}} \|p\|_{1,\Omega} |||(\mathbf{e}_h, y_h)|||,$$

Finally, for the inconsistent term have in (29) we have

$$\begin{aligned}
j_p(\pi_h p, y_h) &\leq j_p(\pi_h p, \pi_h p)^{\frac{1}{2}} j_p(y_h, y_h)^{\frac{1}{2}} \\
&\leq j_p(\pi_h p, \pi_h p)^{\frac{1}{2}} |||(\mathbf{e}_h, y_h)|||
\end{aligned}$$

and we conclude by an application of Lemma 4.3 to estimate $j_p(\pi_h p, \pi_h p)^{\frac{1}{2}}$. $\square$

The following corollary is an immediate consequence of the above convergence analysis.

Corollary 4.1 *Let $(\mathbf{u}, p) \in [H^2(\Omega)]^d \times H^1(\Omega)$ be the solution of (1), $(\mathbf{u}_h, p_h) \in W_h^k$ be the solution of (6) and $y_h \stackrel{\text{def}}{=} p_h - \pi_h p$. Then, there exists a constant $C > 0$ independent of h such that*

$$\begin{aligned}
j_p(y_h, y_h)^{\frac{1}{2}} &\leq C \max_{K \in \mathcal{T}_h} \left\{ \min\left\{ \|\boldsymbol{\beta}\|^{-1}_{0,\infty,K}, \frac{h_K}{\nu} \right\} \right\}^{\frac{1}{2}} h^{\frac{1}{2}} \|p\|_{1,\Omega} \\
&\qquad + C\left(1 + \max\{\nu, \|\boldsymbol{\beta}\|_{0,\infty,\Omega} h\}^{\frac{1}{2}}\right) h \|\mathbf{u}\|_{2,\Omega}, \\
j_p(p_h, p_h) &\leq C j_p(y_h, y_h).
\end{aligned}$$

Now we state a convergence result for the pressure.

Theorem 4.4 *Let $(\mathbf{u}, p) \in [H^2(\Omega)]^d \times H^1(\Omega)$, be solution of (1) and $(\mathbf{u}_h, p_h)$ be the corresponding finite element solution of (6) then there holds*

$$\|p - p_h\|_{0,\Omega} \leq C \max_{K \in \mathcal{T}_h} \left\{ \min\left\{ \|\boldsymbol{\beta}\|^{-1}_{0,\infty,K}, \frac{h}{\nu} \right\} \right\}^{\frac{1}{2}} h^{\frac{1}{2}} \left(\|p\|_{1,\Omega} + \|u\|_{2,\Omega} \right).$$

Proof. In the same spirit as the proof of Theorem 4.2. Using Corollary 4.1 for proving that the discrete pressure gradient jumps converges. □

The following theorem gives an optimal L^2–error estimate for velocity when the local Reynolds number is low.

Theorem 4.5 *Let* $(\mathbf{u}, p) \in [H^2(\Omega)]^d \times H^1(\Omega)$ *be the solution of* (1) *and* $(\mathbf{u}_h, p_h) \in W_h^k$ *be the solution of* (6). *Assume that*

$$\|\boldsymbol{\beta}\|_{0,\infty,\Omega} h \leq \nu, \tag{30}$$

and that the solution (φ, ψ) *of the adjoint problem*

$$\begin{cases} \sigma\varphi - \boldsymbol{\beta} \cdot \nabla\varphi - 2\nu\nabla \cdot \boldsymbol{\varepsilon}(\varphi) - \nabla\psi = \mathbf{u} - \mathbf{u}_h \quad \text{in} \quad \Omega, \\ \nabla \cdot \varphi = 0 \quad \text{in} \quad \Omega, \\ \varphi = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{31}$$

belongs to $[H^2(\Omega)]^d \times [H^1(\Omega)]$ *and satisfies*

$$\|\varphi\|_{2,\Omega} + \|\psi\|_{1,\Omega} \leq C\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}. \tag{32}$$

Then there exists a constant $C > 0$ *independent of* h *such that*

$$\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} \leq Ch^2\big(\|\mathbf{u}\|_{2,\Omega} + \|p\|_{1,\Omega}\big).$$

Proof. Multiplying the first equation of (31) by $\mathbf{u} - \mathbf{u}_h$ and the second by $-(p - p_h)$, integrating by parts and using the modified Galerkin orthogonality (28), if follows that

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}^2 &= a_h(\mathbf{u} - \mathbf{u}_h, \varphi) + b_h(p - p_h, \varphi) - b_h(\psi, \mathbf{u} - \mathbf{u}_h) \\ &= \underbrace{a_h(\mathbf{u} - \mathbf{u}_h, \varphi - \pi_h\varphi) + b_h(p - p_h, \varphi - \pi_h\varphi) - b_h(\psi - \pi_h\psi, \mathbf{u} - \mathbf{u}_h)}_{\mathrm{I}} \\ &\quad + \underbrace{j_{\mathbf{u}}(\mathbf{u} - \mathbf{u}_h, \varphi - \pi_h\varphi)}_{\mathrm{II}} + \underbrace{j_p(p_h, \pi_h\psi)}_{\mathrm{III}}. \end{aligned}$$

Following the argument of the proofs of Theorems 4.1 and 4.2, and (30) we get

$$\begin{aligned} \mathrm{I} &\leq |\!|\!|\mathbf{u} - \mathbf{u}_h|\!|\!|_{\mathbf{u}} |\!|\!|\varphi - \pi_h\varphi|\!|\!|_{\mathbf{u}} + |(\mathbf{u} - \mathbf{u}_h, \boldsymbol{\beta} \cdot \nabla(\varphi - \pi_h\varphi))| \\ &\quad - \langle 2\nu\boldsymbol{\varepsilon}(\mathbf{u} - \mathbf{u}_h)\mathbf{n}, \varphi - \pi_h\varphi\rangle_{\partial\Omega} - \langle \mathbf{u} - \mathbf{u}_h, 2\nu\boldsymbol{\varepsilon}(\varphi - \pi_h\varphi)\mathbf{n}\rangle_{\partial\Omega} \\ &\leq C|\!|\!|\mathbf{u} - \mathbf{u}_h|\!|\!|_{\mathbf{u}} |\!|\!|\varphi - \pi_h\varphi|\!|\!|_{\mathbf{u}} + C|\!|\!|\mathbf{u} - \mathbf{u}_h|\!|\!|_{\mathbf{u}} h\|\varphi\|_{2,\Omega} \\ &\quad + C\|p - p_h\|_{0,\Omega} h\|\varphi\|_{2,\Omega} + C|\!|\!|\mathbf{u} - \mathbf{u}_h|\!|\!|_{\mathbf{u}} \|h\|\psi\|_{1,\Omega}. \end{aligned}$$

Using Cauchy-Schwarz and (16)-(18) one obtains

$$\begin{aligned} \mathrm{II} &\leq j_{\mathbf{u}}(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{u}_h)^{\frac{1}{2}} j_{\mathbf{u}}(\varphi - \pi_h\varphi, \varphi - \pi_h\varphi)^{\frac{1}{2}} \\ &\leq C\|\boldsymbol{\beta}\|_{0,\infty,\Omega} |\!|\!|\mathbf{u} - \mathbf{u}_h|\!|\!|_{\mathbf{u}} h^{\frac{3}{2}} \|\varphi\|_{2,\Omega}. \end{aligned}$$

Finally, using Corollary 4.3 and (30), for the last term we have

$$\begin{aligned}\mathrm{III} &\leq j_p(p_h,p_h)^{\frac{1}{2}} j_p(\pi_h\psi,\pi_h\psi)^{\frac{1}{2}}\\ &\leq C j_p(p_h,p_h)^{\frac{1}{2}} \max_{K\in\mathcal{T}_h}\left\{\min\left\{\|\boldsymbol{\beta}\|_{0,\infty,K}^{-1},\frac{h_K}{\nu}\right\}\right\}^{\frac{1}{2}} h^{\frac{1}{2}}\|\psi\|_{1,\Omega}\\ &\leq \frac{C}{\nu} j_p(p_h,p_h)^{\frac{1}{2}} h\|\psi\|_{1,\Omega}\end{aligned}$$

The proof concludes by combining the above estimations with (30), Theorems 4.3 and 4.4, Corollary 29, and the assumed regularizing behavior (32). □

Let us sum up the results obtained so far. Theorem 4.1 and Theorem 4.2 show that when the local Reynolds number is high and the solution is regular we will enjoy an optimal $O(h^{k+\frac{1}{2}})$ order convergence of the error in the L^2 norm for the velocity and the pressure. For solutions that are less regular Theorem 4.3 shows that when the pressure is in $H^1(\Omega)$ and the velocity in $[H^2(\Omega)]^d$ we get an optimal $O(h)$ estimate in the energy norm when the local Reynolds number is low, but a suboptimal estimate of $O(h^{\frac{1}{2}})$ when the local Reynolds number is high. This is due to the fact that the inconsistencies in the pressure stabilization pollutes the energy norm estimate for the velocities.

4.2.1 Alternative stabilization parameters for the high Reynolds number regime

A way of obtaining an estimate which is optimal in both the high order regime and the low order regime when the pressure is only H^1 is to modify the stabilization parameters to give control in a stronger norm in the high Reynolds number regime. In this section we propose such an alternative set of stabilization parameters. Since the techniques of proof are identical to those of the previous section we only outline the proofs. First we drop the dependence of the local Reynolds number in the pressure stabilization and instead increase the stabilization on the incompressibility condition. This way the triple norm will be dominated by the H^1-norm in the low local Reynolds number regime and by the H_{div}-norm in the high Reynolds number regime. Since $\mathbf{u}\in[H^2(\Omega)]^d$ and $p\in H^1(\Omega)$ we must relax the stabilization of the pressure in order to account for the inconsistency. To obtain optimality when estimating the pressure velocity coupling terms the stronger control of the incompressibility condition is required. With this aim, we consider here the modified stabilization terms, for the stabilization of the incompressibility and the pressure

$$\begin{aligned}&\sum_{K\in\mathcal{T}_h}\gamma_{\text{div}} h_K\int_{\partial K}[\![\nabla\cdot\mathbf{u}_h]\!][\![\nabla\cdot\mathbf{v}_h]\!]\,\mathrm{d}s,\\ &\sum_{K\in\mathcal{T}_h}\gamma_p h_K^3\int_{\partial K}[\![\nabla p_h]\!]\cdot[\![\nabla q_h]\!]\,\mathrm{d}s.\end{aligned}\tag{33}$$

We also modify the scaling with h of the last boundary term of (8) taking now

$$\left\langle\gamma_{\mathbf{n}} h^{-1}\mathbf{u}_h\cdot\mathbf{n},\mathbf{v}_h\cdot\mathbf{n}\right\rangle_{\partial\Omega}.\tag{34}$$

First note that the inconsistent pressure jump term has the right order. This follows by Lemma 4.3 with the new stabilizing parameter. Again, since p does not have the regularity required for $j_p(p,p)$ to make sense the triple norm must be redefined in a similar fashion as in Section 4.2. Hence we use a triple norm without the pressure penalty term, namely,

$$\begin{aligned}|||\mathbf{v}_h|||_{\mathbf{u}}^2 = \|\sigma^{\frac{1}{2}}\mathbf{v}_h\|_{0,\Omega}^2 + \|\nu^{\frac{1}{2}}\nabla\mathbf{v}_h\|_{0,\Omega}^2 + j_{\mathbf{v}}(\mathbf{v}_h,\mathbf{v}_h) + \||\boldsymbol{\beta}\cdot\mathbf{n}|^{\frac{1}{2}}\mathbf{v}_h\|_{0,\partial\Omega_{\text{in}}}^2 \\ + \|(\gamma_\nu\nu)^{\frac{1}{2}}h^{-\frac{1}{2}}\mathbf{v}_h\|_{0,\partial\Omega}^2 + \|\gamma_{\mathbf{n}}^{\frac{1}{2}}h^{-\frac{1}{2}}\mathbf{v}_h\cdot\mathbf{n}\|_{0,\partial\Omega}^2,\end{aligned}$$

for all $\mathbf{v}_h \in [V_h^k]^d$.

Lemma 4.5 *Let $\mathbf{u} \in [H^2(\Omega)]^d$ then there holds*

$$|||\mathbf{u}-\pi_h\mathbf{u}|||_{\mathbf{u}} \leq \left[\sigma^{\frac{1}{2}}h^2 + \max\{\|\boldsymbol{\beta}\|_{0,\infty,\Omega}h,\nu\}^{\frac{1}{2}}h + \gamma_{\text{div}}h\right]\|\mathbf{u}\|_{2,\Omega},$$

Proof. Identical to the proof of Lemma 4.2, taking into account the modified stabilization parameter in the incompressibility stabilization. □

Note that the approximation order of Lemma 4.5 is always h. In the high Reynolds number regime, we show in the following lemma that the triple norm gives control of $\|\nabla\cdot\mathbf{u}_h\|_{0,\Omega}$. Hence the approximability result is optimal.

Lemma 4.6 *Using the modified stabilizing terms* (33)*, the discrete formulation* (6) *satisfies*

$$\|\nabla\cdot\mathbf{u}_h\|_{0,\Omega} \leq |||(\mathbf{u}_h,p_h)|||_{\mathbf{u}}.$$

Proof. Testing with $(\mathbf{v}_h,q_h) = (\mathbf{0},\pi_h\nabla\cdot\mathbf{v}_h)$ yields

$$\|\nabla\cdot\mathbf{u}_h\|_{0,\Omega}^2 \leq \|\nabla\cdot\mathbf{u}_h - \pi_h\nabla\cdot\mathbf{u}_h\|_{0,\Omega}^2 + j_p(p_h,\pi_h\nabla\cdot\mathbf{u}_h) + \langle\pi_h\nabla\cdot\mathbf{u}_h,\mathbf{u}_h\cdot\mathbf{n}\rangle_{\partial\Omega}.$$

For the first term, we have

$$\|\nabla\cdot\mathbf{u}_h - \pi_h\nabla\cdot\mathbf{u}_h\|_{0,\Omega}^2 \leq \|\nabla\cdot\mathbf{u}_h - \pi_h^*\nabla\cdot\mathbf{u}_h\|_{0,\Omega}^2 \leq j_{\mathbf{u}}(\mathbf{u}_h,\mathbf{u}_h).$$

For the second term, applying Cauchy-Schwarz inequality, a trace inequality and an inverse inequality, we obtain

$$\begin{aligned}j_p(p_h,\pi_h\nabla\cdot\mathbf{u}_h) &\leq j_p(p_h,p_h)^{\frac{1}{2}}j_p(\pi_h\nabla\cdot\mathbf{u}_h,\pi_h\nabla\cdot\mathbf{u}_h)^{\frac{1}{2}} \\ &\leq \epsilon^{-1}j_p(p_h,p_h) + \epsilon C\|\nabla\cdot\mathbf{u}_h\|^2,\end{aligned}$$

with $\epsilon > 0$ constant to be fixed later on. Finally, for the boundary term, we obtain in the same fashion

$$\langle\pi_h\nabla\cdot\mathbf{u}_h,\mathbf{u}_h\cdot\mathbf{n}\rangle_{\partial\Omega} \leq \epsilon^{-1}\|\gamma_{\mathbf{n}}h^{-\frac{1}{2}}\mathbf{u}_h\cdot\mathbf{n}\|_{\partial\Omega}^2 + \epsilon C\|\nabla\cdot\mathbf{u}_h\|^2.$$

We conclude proof by choosing $\epsilon > 0$ small enough. □

Theorem 4.6 *Let* $(\mathbf{u},p) \in [H^2(\Omega)]^d \times H^1(\Omega)$ *be the solution of* (1) *and* $(\mathbf{u}_h,p_h) \in W_h$ *be the solution of* (6) *then*

$$\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{u}} \leq Ch\|p\|_{1,\Omega} \\ + C\left[\left(\sigma^{\frac{1}{2}} + \frac{\|\boldsymbol{\beta}\|_{1,\infty,\Omega}}{\sigma^{1/2}}\right)h^2 + \max\{\|\boldsymbol{\beta}\|_{0,\infty,\Omega}h,\nu\}^{\frac{1}{2}}h + \gamma_{\mathrm{div}}h\right]\|\mathbf{u}\|_{2,\Omega}.$$

Proof. We follow the proof of Theorem 4.3 taking into account the results of Lemmas 4.6 and 4.3. Thus, we have

$$\begin{aligned} b_h(y^\pi,\mathbf{e}_h) &= -(y^\pi,\nabla\cdot\mathbf{e}_h) + \langle y^\pi,\mathbf{e}_h\cdot\mathbf{n}\rangle_{\partial\Omega} \\ &\leq \|y^\pi\|_{0,\Omega}\|\nabla\cdot\mathbf{e}_h - \pi_h^*\nabla\cdot\mathbf{e}_h\|_{0,\Omega} + C\|y^\pi h^{\frac{1}{2}}\|_{0,\partial\Omega}\|\gamma_{\mathbf{n}}^{\frac{1}{2}}h^{-\frac{1}{2}}\mathbf{e}_h\cdot\mathbf{n}\|_{0,\partial\Omega} \\ &\leq Ch\|p\|_{1,\Omega}\|(\mathbf{e}_h,y_h)\|, \\ j_p(\pi_hp,y_h) &\leq j_p(\pi_hp,\pi_hp)^{\frac{1}{2}}j_p(y_h,y_h)^{\frac{1}{2}} \\ &\leq j_p(\pi_hp,\pi_hp)^{\frac{1}{2}}\|(\mathbf{e}_h,y_h)\| \\ &\leq Ch\|p\|_{1,\Omega}\|(\mathbf{e}_h,y_h)\|, \end{aligned}$$

which allow us to obtained the desired estimate. □

An immediate consequence of the above convergence analysis is the following corollary

Corollary 4.2 *For* $y_h = p_h - \pi_hp$ *there holds*

$$\begin{aligned} j_p(y_h,y_h)^{\frac{1}{2}} &\leq Ch\|p\|_{1,\Omega} + C(\nu^{\frac{1}{2}}h + h)\|\mathbf{u}\|_{2,\Omega}, \\ j_p(p_h,p_h) &\leq Cj_p(y_h,y_h). \end{aligned}$$

Theorem 4.7 *Let* $(\mathbf{u},p) \in [H^2(\Omega)]^d \times H^1(\Omega)$, *be solution of* (1)*) and* $(\mathbf{u}_h,p_h)$ *be the corresponding finite element solution of* (6) *then there holds*

$$\|p-p_h\| \leq Ch(\|p\|_{1,\Omega} + \|u\|_{2,\Omega})$$

Proof. In the same spirit as the proof of Theorem 4.2, using Corollary 4.2 to control the discrete pressure gradient jumps. □

5 Edge stabilization and large eddy simulation

The method presented above has some common features with variational multiscale methods for large eddy simulations (VMS) as introduced in [24]. Even though our main objective is to formulate a finite element method which is stable for all Reynolds numbers the interpretation of the edge stabilization method using Nitsche type weakly imposed boundary condition as method for LES is interesting. However unlike the VMS where two scales V_h and V_H are considered, in our case the finite element space V_h represents the only resolved scale and the "turbulent" viscosity acts only on the scales that are not resolved on V_h. Note also that the weak boundary conditions will act as slip boundary conditions in the high Reynolds number regime and as no-slip conditions when the boundary layers are resolved very much like one would like them to do in LES. Recently a VMS was proposed using a projection method framework [25]. The idea was to work on two different scales V_h and V_H and introduce the turbulence model only on the finer scales represented by $V_h \setminus V_H$. The formulation of [25] essentially takes the form of a standard Galerkin formulation for $\mathbf{u}_h$ supplemented with the turbulent viscosity acting only on the fine scales in the form of an additional term

$$(\nu_T(I - P_H)\boldsymbol{\varepsilon}(\mathbf{u}_h), (I - P_H)\boldsymbol{\varepsilon}(\mathbf{v}_h)), \tag{35}$$

where P_H is some map from fine scales to coarse scales. Comparing now with the edge stabilization method we would choose $H = h$ and thus make the turbulent viscosity act only on the scales that are not resolved on the space V_h. Applying Lemma 3.1 we immediately get an interior penalty interpretation of the term (35), with $P_H \stackrel{\text{def}}{=} \pi_h^*$,

$$\|\nu_T^{\frac{1}{2}}(I - \pi_h^*)\boldsymbol{\varepsilon}(\mathbf{u}_h)\|_\Omega^2 \leq \sum_{K\in\mathcal{T}_h} \int_{\partial K} \nu_T h_K [\![\boldsymbol{\varepsilon}(\mathbf{u}_h)]\!] : [\![\boldsymbol{\varepsilon}(\mathbf{u}_h)]\!] \,\mathrm{d}s$$

and we conclude that a possible subgrid modeling term would be

$$j_T(\mathbf{u}_h, \mathbf{v}_h) = \sum_{K\in\mathcal{T}_h} \int_{\partial K\setminus\partial\Omega} \nu_T h_K [\![\boldsymbol{\varepsilon}(\mathbf{u}_h)]\!] : [\![\boldsymbol{\varepsilon}(\mathbf{v}_h)]\!] \,\mathrm{d}s$$

where the choice of ν_T now is a modeling issue. It should be noted that the choice $\nu_T = \gamma h_K$ gives us a term which is asymptotically equivalent to the edge stabilization operator using the whole gradient. This means that any choice of ν_T with higher order in h_K will be negligible compared to the effect of the stabilizing terms. For instance the Smagorinsky turbulence model given by $\nu_T = Ch_K^2|\boldsymbol{\varepsilon}(\mathbf{u}_h)|$ applied only on the fine scales must be expected to be completely dominated by the stabilizing terms except when $|\nabla \mathbf{u}_h| \approx h_K^{-1}$. Another interesting observation in this framework is that from polynomial degree three there exists a C^1 subspace of V_h with approximation properties (starting with the reduced Hsieh-Clough-Tocher space, see [13]). It follows that the solution may be decomposed into one C^1 part which is untouched by the stabilizing terms and another C^0 part which is penalized. We conclude that the method enjoys the scale separation property characteristic for variational multiscale methods as proposed in [24] by polynomial order rather than by hierarchic meshes. For a more detailed discussion concerning (local) projection stabilized finite elements and large eddy simulation see [3].

6 Numerical results

In this section we report several numerical experiments that shows the good convergence properties of our stabilized finite element method. In particular, we recover the convergence rates obtained in Section 4.

6.1 Convergence study in a 2D case with small viscosity

Let $\lambda = (\nu^{-1} - (\nu^{-2} + 16\pi 2)^{\frac{1}{2}})/2$. Then the exact solution to (1) is given by (see [27])

$$\begin{aligned} &u_1(x_1,x_2) = 1 - e^{\lambda x_1} \cos 2\pi\, x_2,\\ &u_2(x_1,x_2) = \tfrac{\lambda x_1}{2\pi} e^{\lambda} \sin 2\pi\, x_2,\\ &p = \tfrac{1}{2} e^{2\lambda x} + C, \end{aligned}$$

with $\boldsymbol{\beta} = u$, $\sigma = 0$ and a right hand side matching the exact solution. In our examples, we also chose C to give zero mean pressure. We solved this problem approximatively on $\Omega = (-1/2, 3/2) \times (0, 2)$, using stability parameters using stability parameters $\gamma_{\boldsymbol{\beta}} = \gamma_p = \gamma = 1/100$ and $\gamma_{\boldsymbol{\beta}} = 1/10$. We present the solutions on two different meshes in figure 2 and 1 In figure 3 we show the convergence for $\nu = 10^{-4}$.

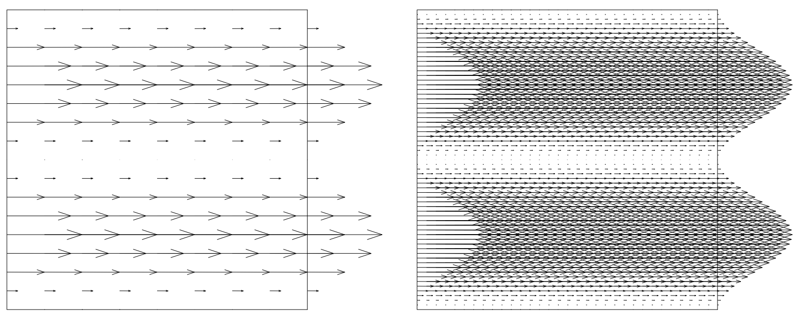

Figure 1: Left: coarse mesh velocities. Right: fine mesh velocities

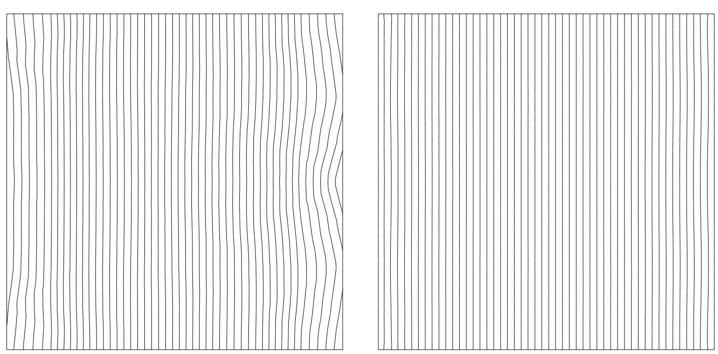

Figure 2: Left: coarse mesh pressure. Right: fine mesh pressure

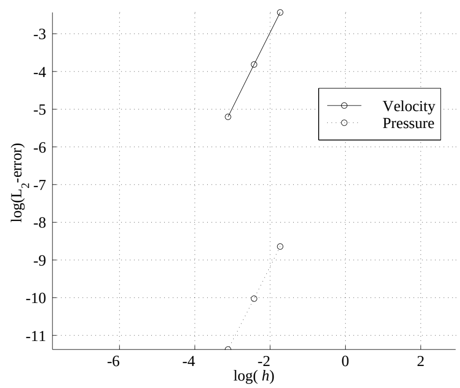

Figure 3: Convergence history for $\nu = 10^{-4}$

6.2 Convergence study in a 3D case with small viscosity

We consider now problem (1) in 3D with non-homogeneous boundary conditions. The right hand side $\mathbf{f}$ and the boundary data are chosen in order to ensure that the exact solution of (1) is given by the following expression [18]:

$$\begin{aligned} u_1(x_1,x_2,x_3) &= be^{a(x_1-x_3)+b(x_2-x_3)} - ae^{a(x_3-x_2)+b(x_1-x_2)}, \\ u_2(x_1,x_2,x_3) &= be^{a(x_2-x_1)+b(x_3-x_2)} - ae^{a(x_1-x_3)+b(x_2-x_3)}, \\ u_3(x_1,x_2,x_3) &= be^{a(x_3-x_2)+b(x_1-x_2)} - ae^{a(x_2-x_1)+b(x_3-x_1)}, \\ p(x_1,x_2,x_3) &= (a^2+b^2+ab)\Big[e^{a(x_1-x_2)+b(x_1-x_3)} \\ &\quad + e^{a(x_2-x_3)+b(x_2-x_1)} + e^{a(x_3-x_1)+b(x_3-x_2)}\Big]. \end{aligned} \tag{36}$$

with $\boldsymbol{\beta} = \mathbf{u}$, $\sigma = 1$, $\nu = 10^{-4}$, $a = b = 0.75$ and $\Omega = (0,1)^3$ the unit cube.

The resulting continuous problem was solved approximatively using the stabilized discrete formulation (6), however the boundary conditions were strongly enforced. All numerical tests have been performed using conforming linear and quadratic finite elements for velocity and pressure, namely P_1/P_1 and P_2/P_2 (implemented in a 3D research code [19]). The stabilization parameters involved in the jumps terms (10) and (11) were chosen as

$$\gamma_p = \gamma_{\boldsymbol{\beta}} = \gamma_{\text{div}} = \begin{cases} \dfrac{1}{8} & \text{if} \quad k = 1, \\ \dfrac{1}{32} & \text{if} \quad k = 2, \end{cases}$$

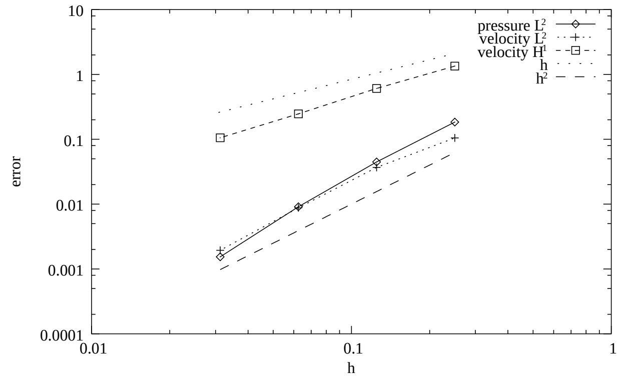

Figure 4: Convergence history: linear elements ($k = 1$)

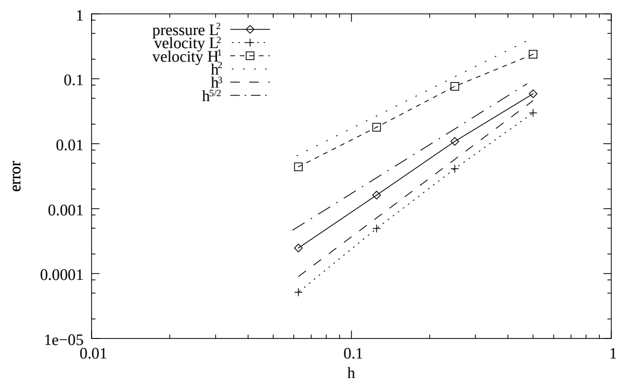

Figure 5: Convergence history: quadratic elements ($k = 2$)

In figures 4 and 5 we show, respectively, the velocity and pressure convergence histories for $k = 1$ and $k = 2$. Note that, in both cases, the numerical solution exhibits optimal convergence order, hence, in agreement with Theorems 4.1 and 4.2.

We show, in figure 7 the pressure contours in two different meshes (which are depicted in figure 6) using linear elements. No spurious pressure oscillations are observed. We report in figure 8 the contours of the second component of $\mathbf{u}_h$, u_{h2}, with $\gamma_{\boldsymbol{\beta}} = 1/8$ and $\gamma_{\boldsymbol{\beta}} = 0$, on the cutting plane $x = 0.5$. Although the exact solution is smooth the plot of the unstabilized solution (right) exhibits spurious oscillations. Note that the spurious velocity oscillations (right) are completely controlled by the streamline-derivative jumps (left).

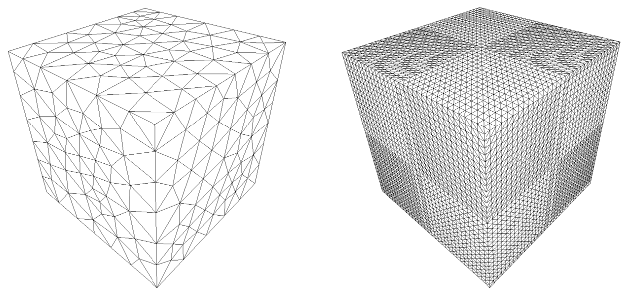

Figure 6: Coarse mesh (2929 tetrahedra) and fine mesh (196608 tetrahedra)

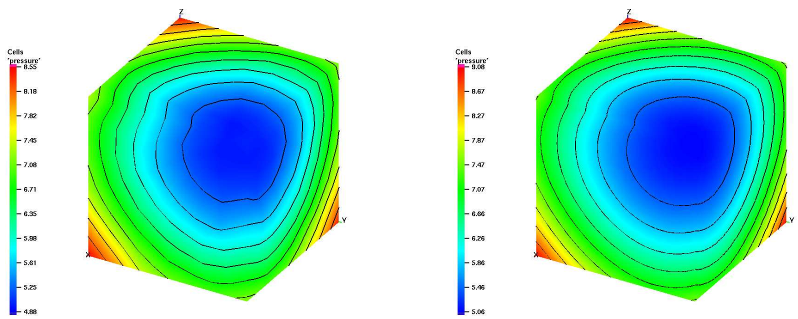

Figure 7: Pressure contours: coarse mesh (left) and fine mesh (right)

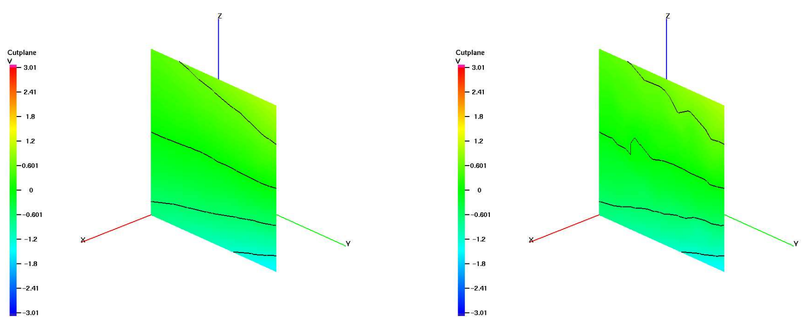

Figure 8: Velocity (u_{h2}) contours on a cutting plane: stabilized (left), with $\gamma_{\boldsymbol{\beta}} = 0$ (right)

In the sequel we will replace, in (36), the expression for the pressure by

$$p(x_1, x_2, x_3) = \begin{cases} 2x_2 & \text{if} \quad 0 \leq x_2 \leq \dfrac{1}{2}, \\ 2(1 - x_2) & \text{if} \quad \dfrac{1}{2} \leq x_2 \leq 1. \end{cases}$$

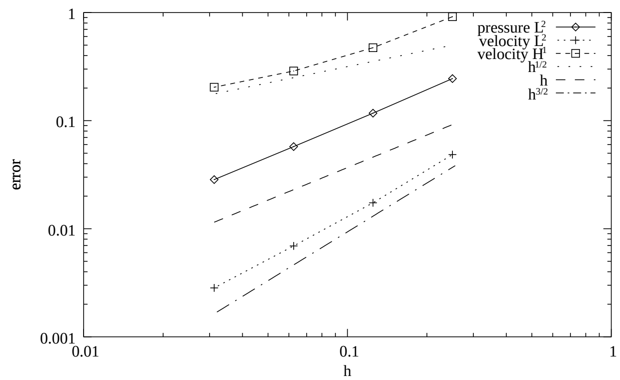

Figure 9: Convergence history: linear elements, non-smooth pressure

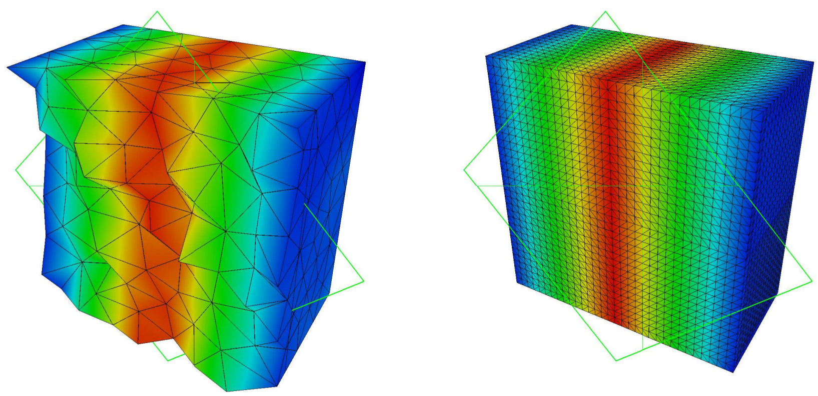

Figure 10: Cutting plane pressure: coarse mesh and fine mesh

Clearly, this function satisfies $p \in H^1(\Omega)$ but does not belong to $H^2(\Omega)$, so that we are in the framework of Section 4.2. Figure 10 shows the pressure contours in a cut of a coarse and a fine mesh. Once more no spurious pressure oscillations are observed. Figure 9 shows the velocity and pressure convergence histories using linear elements. We get the sub-optimal $O(h^{\frac{1}{2}})$ order for the velocity in the H^1–norm in the case of high local Reynolds numbers, in agreement with Theorem 4.3. The L^2-norm of the velocities on the other hand is still not far from the quasi optimal $O(h^{\frac{3}{2}})$ convergence order. As expected, when the local Reynolds is low (for instance $\nu = 0.1$), we recover the optimal $O(h)$, see figure 11. In addition, as predicted in Theorem 4.5, we notice that the convergence order for the velocity in the L^2–norm is $O(h^2)$.

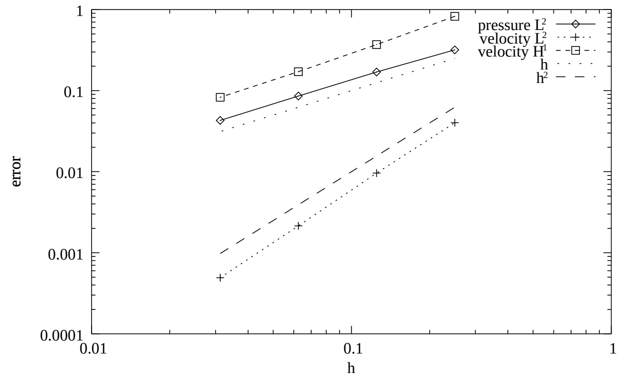

Figure 11: Convergence history: linear elements, non-smooth pressure, low local Reynolds number

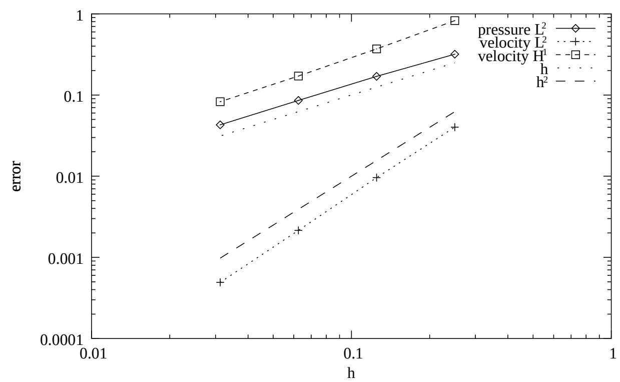

Figure 12: Convergence history: linear elements, non-smooth pressure, alternative parameters

Finally, figure 12 shows that the alternative stabilization parameters proposed in Section 4.2.1 give optimal convergence order for the velocity and the pressure.

7 Conclusion

In this paper, we propose a stabilized finite element method for the linearized incompressible Navier-Stokes (Oseen's) equations. The method allows equal velocity-pressure interpolation and gives optimal control of the streamline derivative of the velocity. The stability properties of the method are based on an interior penalty term giving L^2-control of the jump of the gradient over interior element edges. We show that such a stabilization operator may be used to control all the non-symmetric first order terms of the Oseen's equations and that they give control only of the part of the operator that is not in the finite element space. In this sense the proposed method is a minimal stabilized method (see [6]).

The present work extends the results reported in [11, 10] to the Oseen's equations using equal order interpolation and finite element spaces of any polynomial order.

One of the main properties of the method is that it overcomes some undesirable features of SUPG-like stabilized methods. Thus, no artificial velocity-pressure couplings (pressure and velocity stabilizations are independent [8]) nor artificial boundary conditions are introduced, and the mass matrix can be lumped for low order elements. This is thanks to the fact that our stabilization does not rely on the addition of low order element residuals of least squares type. The price to pay is some added couplings in the matrix due to jump coupling. The bandwidth of the system matrix doubles (with respect to SUPG) in two space dimensions and triples in three space dimensions.

The convergence analysis shows that the method has (quasi) optimal convergence properties in both in the L^2-norm and in the energy norm when the solution is sufficiently regular or the local Reynolds number is low. When physical realistic regularities are considered ($p \in H^1(\Omega)$) and the local Reynolds number is high, the convergence may become sub-optimal $O(h^{\frac{1}{2}})$, due to the inconsistencies in the pressure stabilization. Optimal convergence order may be recovered by using an alternative set of stabilization parameters. Some numerical results illustrated the theoretical results and showed very good performance in all regimes. In particular we observe that in the high Reynolds number regime the scheme degenerates to the theoretical $O(h^{k+\frac{1}{2}})$ convergence in the L^2 norm predicted by the theory only in the case where the pressure is only H^1 and where the theoretical prediction is $O(h^{\frac{1}{2}})$. However recalling the results of [7] on Peterson meshes it seems likely that the results obtained are sharp. In a forthcoming paper we will discuss how to minimize the matrix bandwidth while retaining optimal convergence properties and extend the analysis to the time-dependent problem.

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A Proof of Theorem 3.1

Let $(\mathbf{u}_h, p_h) \in W_h^k$. Following [21] there exists $\mathbf{v}_p \in [H_0^1(\Omega)]^d$ such that

$$\begin{aligned} \nabla \cdot \mathbf{v}_p &= -p_h, \\ \|\mathbf{v}_p\|_{1,\Omega} &\le c_1 \|p_h\|_{0,\Omega}, \end{aligned} \tag{37}$$

with $c_1 > 0$ depending only on Ω. From (7) we have

$$\begin{aligned}\mathbf{A}\big((\mathbf{u}_h, p_h), (\pi_h \mathbf{v}_p, 0)\big) = a_h(\mathbf{u}_h, \pi_h \mathbf{v}_p) + j_{\mathbf{u}}(\mathbf{u}_h, \pi_h \mathbf{v}_p) \\ + b_h(p_h, \pi_h \mathbf{v}_p)\end{aligned} \tag{38}$$

From (9), since $\mathbf{v}_p \in [H_0^1(\Omega)]^d$, $h \leq 1$, $\nu \leq 1$, and using (37) and (13) we get

$$\begin{aligned}b_h(p_h, \pi_h \mathbf{v}_p) &= -(p_h, \nabla \cdot \pi_h \mathbf{v}_p) + \langle p_h, \pi_h \mathbf{v}_p \cdot \mathbf{n} \rangle_{\partial\Omega} \\ &= -(p_h, \nabla \cdot (\pi_h \mathbf{v}_p - \mathbf{v}_p)) + \langle p_h, (\pi_h \mathbf{v}_p - \mathbf{v}_p) \cdot \mathbf{n} \rangle_{\partial\Omega} \\ &\quad + \|p_h\|_{0,\Omega}^2 \\ &= (\nabla p_h, \pi_h \mathbf{v}_p - \mathbf{v}_p) + \|p_h\|_{0,\Omega}^2 \\ &= (\nabla p_h - \pi_h^*(\nabla p_h), \pi_h \mathbf{v}_p - \mathbf{v}_p) + \|p_h\|_{0,\Omega}^2 \\ &\geq -\|\nabla p_h - \pi_h^*(\nabla p_h)\|_{0,\Omega} \|\pi_h \mathbf{v}_p - \mathbf{v}_p\|_{0,\Omega} + \|p_h\|_{0,\Omega}^2 \\ &\geq -C \max\{\nu, \|\boldsymbol{\beta}\|_{1,\infty,\Omega}\}^{\frac{1}{2}} h^{\frac{1}{2}} j_p^{\frac{1}{2}}(p_h, p_h) \|p_h\|_{0,\Omega} + \|p_h\|_{0,\Omega}^2 \\ &\geq -\frac{C_1}{\alpha_1} j_p(p_h, p_h) + (1 - C_1 \alpha_1) \|p_h\|_{0,\Omega}^2 \\ &\geq -\frac{C_1}{\alpha_1} \|\!|\!|(\mathbf{u}_h, p_h)|\!|\!|_{\mathbf{u}}^2 + (1 - C_1 \alpha_1) \|p_h\|_{0,\Omega}^2.\end{aligned} \tag{39}$$

with α_1 a positive constant to be fixed later on.

Now, combining (37) with the argument used to estimate (26) in the proof of Theorem 4.2, it follows that

$$\begin{aligned}a_h(\mathbf{u}_h, \pi_h \mathbf{v}_p) + j_{\mathbf{u}}(\mathbf{u}_h, \pi_h \mathbf{v}_p) &\geq -C \|\!|\!|(\mathbf{u}_h, 0)|\!|\!| \|\mathbf{v}_p\|_{1,\Omega} \\ &\geq -C_2 \|\!|\!|(\mathbf{u}_h, p_h)|\!|\!| \|p_h\|_{0,\Omega} \\ &\geq -\frac{C_2}{\alpha_2} \|\!|\!|(\mathbf{u}_h, p_h)|\!|\!|_{\mathbf{u}}^2 - \alpha_2 C_2 \|p_h\|_{0,\Omega}^2.\end{aligned} \tag{40}$$

with α_2 a positive constant to be fixed later on.

Thus, from (38), (39) and (40) one gets

$$\begin{aligned}\mathbf{A}\big((\mathbf{u}_h, p_h), (\pi_h \mathbf{v}_p, 0)\big) \geq -\left(\frac{C_1}{\alpha_1} + \frac{C_2}{\alpha_2}\right) \|\!|\!|(\mathbf{u}_h, p_h)|\!|\!|_{\mathbf{u}}^2 \\ + (1 - C_1 \alpha_1 - C_2 \alpha_2) \|p_h\|_{0,\Omega}^2\end{aligned} \tag{41}$$

Hence, if α_1, α_2 are chosen small enough we have

$$\mathbf{A}\big((\mathbf{u}_h, p_h), (\pi_h \mathbf{v}_p, 0)\big) \geq -C_3 \|\!|\!|(\mathbf{u}_h, p_h)|\!|\!|^2 + C_3 \|p_h\|_{0,\Omega}^2, \tag{42}$$

with $C_3 > 0$ constant independent of h and ν.

Now, by setting

$$(\mathbf{v}_h, q_h) \stackrel{\text{def}}{=} (\mathbf{u}_h + \alpha\pi_h \mathbf{v}_p, p_h), \quad \alpha > 0,$$

using the bilinearity of $\mathbf{A}$, Lemma 3.2 and (42), and if α is chosen small enough we obtain

$$\begin{aligned}
\mathbf{A}\big((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)\big) &= \mathbf{A}\big((\mathbf{u}_h, p_h), (\mathbf{u}_h, p_h)v\big) + \alpha\mathbf{A}\big((\mathbf{u}_h, p_h), (\pi_h \mathbf{v}_p, 0)\big) \\
&\geq C|||(\mathbf{u}_h, p_h)|||^2 - \alpha C_3|||(\mathbf{u}_h, p_h)|||^2 + \alpha C_3\|p_h\|_{0,\Omega}^2 \\
&\geq C_4|||(\mathbf{u}_h, p_h)|||_*^2.
\end{aligned} \tag{43}$$

with $C_4 > 0$ independent of h and ν. Finally, from (37), it follows that

$$|||(\pi_h \mathbf{v}_p, 0)||| \leq C\|p_h\|_{0,\Omega},$$

and therefore

$$\begin{aligned}
|||(\mathbf{v}_h, q_h)|||_*^2 &= |||(\mathbf{u}_h + \alpha\pi_h \mathbf{v}_p, p_h)|||^2 + \|p_h\|_{0,\Omega}^2 \\
&\leq 2|||(\mathbf{u}_h, p_h)|||^2 + 2\alpha^2|||(\pi_h \mathbf{v}_p, 0)|||^2 + \|p_h\|_{0,\Omega}^2 \\
&\leq C_5|||(\mathbf{u}_h, p_h)|||_*^2,
\end{aligned}$$

with $C_5 > 0$ independent of h and ν. The above estimation, combined with (43), completes the proof.

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